Conjugacy classes in locally compact subgroups of S_∞

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HIM Workshop on Homogeneous Structures

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- The automorphism group of a countable first order structure is isomorphic to a closed subgroup of S_∞.
- Every closed subgroup of S_∞ is the automorphism group of a countable first order structure in a countable language. Indeed, any such subgroup is the automorphism group of some Fraïssé limit in a countable language.

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Certainly the converse holds as well.

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Fact

Let $\mathcal{M}$ be a countable L-structure. Then, $Aut(\mathcal{M})$ is locally compact if and only if there exists $w_0, ..., w_n \in M$ with $acl^{\mathcal{M}}_{grp}(w_0, ..., w_n) = M$.

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Moral

Non-discrete t.d.l.c. Polish groups - i.e. non-discrete locally compact subgroups of S_∞ - are interesting mathematical objects which appear to admit general, interesting theorems.

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Moral

Non-discrete t.d.l.c. Polish groups - i.e. non-discrete locally compact subgroups of S_∞ - are interesting mathematical objects which appear to admit general, interesting theorems. Furthermore, their study may be approached via model theoretic and descriptive set theoretic methods.

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Observation

None of the examples are locally compact.

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- Connected locally compact groups cannot have a dense conjugacy class. (K.H. Hofmann)
- So the question reduces to the above.
- The above question indeed asks about non-discrete t.d.l.c. Polish groups.

Theorem (W.)

A non-trivial t.d.l.c. Polish group does not admit a comeagre conjugacy class.

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- These compact open subgroups are *not* necessarily normal.
- These subgroups are *profinite*.
- These subgroups will be the obstruction to a comeagre conjugacy class.

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- Profinite groups behave like finite groups.

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If F is a finite group and $h \in F$, then $h^F \leftrightarrow F/C_F(h)$. In particular, $|h^F| = |F : C_F(h)|$.

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Just consider the map $f \mapsto fhf^{-1}$

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Key Lemma

Let U be a profinite group with normalized Haar measure μ. If $h \in U$ is such that $\mu(h^U) > 0$, then

(i) For all $N \trianglelefteq_o U$, $|C_{U/N}(hN)| \leqslant \frac{1}{\mu(h^U)}$.

(ii) $|C_U(h)| \leqslant \frac{1}{\mu(h^U)}$, and in particular, h is torsion.

Proof.

Take $N \trianglelefteq_o U$.

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- Take a transversal $h, k_1 h k_1^{-1}, ..., k_n h k_n^{-1}$ for $(hN)^{U/N}$ in U/N. Certainly, $h^U \subseteq hN \cup ... \cup k_n h k_n^{-1} N$.

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$$\mu(h^U) \leqslant \mu(hN \cup ... \cup k_n h k_n^{-1} N)$$

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By invariance of the Haar measure, $\mu(h^U) \leqslant |(hN)^{U/N}| \mu(N)$.

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By invariance of the Haar measure, $\mu(h^U) \leqslant |(hN)^{U/N}|\mu(N)$.
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$$\mu(N)|U/N : C_{U/N}(hN)| = \frac{\mu(N)|U/N|}{|C_{U/N}(hN)|} = \frac{1}{|C_{U/N}(hN)|}$$

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- Thus,
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So $|C_{U/N}(hN)| \leqslant \frac{1}{\mu(h^U)}$, and we have (i).
- It follows $|C_U(h)| \leqslant \frac{1}{\mu(h^U)}$ since any finite number of distinct elements are distinct in some finite quotient, and we have (ii).

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(iv) *There is $M > 0$ such that $|C_{U/N}(hN)| \leqslant M$ for all $N \trianglelefteq_o U$.*

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Fact (Rosendal)

There is a non-discrete second countable profinite group with a non-meagre conjugacy class.

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- By uniqueness of Haar measure, h^W is non-null in W.
- The previous theorem implies h^W is open and h is torsion.

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Corollary

Suppose G is t.d.l.c. Polish group and $h \in G$ is periodic. The following are equivalent:

(i) h^G is open.

(ii) h^G is non-meagre.

(iii) h^G is non-null.

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Fact (Akin, Glasner, Weiss [1])

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- By the equivalence of category and measure for conjugacy classes, h^G is open. **Lemma (1)** implies h is torsion. Since h^G is dense, G has exponent $|h|$.
- So every compact open subgroup U is torsion.
- Applying deep theorems of Wilson [4] and Zel'manov [5] on torsion profinite groups now leads to a contradiction.

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Corollary

A non-trivial t.d.l.c. Polish group does not admit a comeagre or co-null conjugacy class.

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  - Wilson's theorem relies on the classification of finite simple groups.
  - We may avoid using them in the special case of ample generics.
- The category/measure equivalence for conjugacy classes in profinite groups seems independently interesting.
- This result supports the suspicion a non-trivial t.d.l.c. group topology imposes non-trivial structure.

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Proposition

A non-trivial t.d.l.c. Polish group G cannot have ample generics.

Proof.

- Suppose not. G may be written as an increasing union of compact open *subgroups*. (Platonov)
- Suppose $(g, h) \in G^{\times 2}$ has a comeagre diagonal conjugacy class.
- By Kuratowski-Ulam, there exists $l \in G$ such that $\forall^* k$, $(l, k) \in (g, h)^G$. So $k^{C_G(l)}$ is comeagre in G.
- l must also have a comeagre conjugacy class.
- By the key lemma, $C_G(l)$ is countable. This is an absurdity.

□

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