Dual Ramsey theorem for trees

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October 2013

Outline of Topics

Introduction

For $n \in \mathbb{N}$, put

$$[n] = \{1, 2, \ldots, n\};$$

in particular,

$$[0] = \emptyset.$$

Dual Ramsey theorem

An k-partition $\mathcal{P}$ is an k-**subpartition** of an l-partition $\mathcal{Q}$ if $\mathcal{P}$ is a coarser partition than $\mathcal{Q}$.

Theorem (Graham–Rothschild)

Let $d > 0$. For each k, l, there exists m such that for each d-coloring of all k-partitions of $[m]$ there exists an l-partition $\mathcal{Q}$ of $[m]$ such that all k-subpartitions of $\mathcal{Q}$ get the same color.

A Ramsey theorem for trees

A **tree** = a finite non-empty tree

For a tree T and $v, w \in T$, let $v \wedge_T w$ be the highest element of T below both v and w.

A function $f\colon S \to T$ is a **morphism** if for all $v, w \in S$,

$$f(v \wedge_S w) = f(v) \wedge_T f(w).$$

$\mathrm{im}_T(v)$ is the set of **immediate successors** of v in T.

A tree T is **ordered** if for each $v \in T$ there is a fixed linear order of $\mathrm{im}_T(v)$. This arrangement induces a lexicographic linear order

$$\leq_T$$

on the whole tree T.

An injection $f\colon S \to T$ is an **embedding** if it is an order preserving tree morphism.

A **copy** of S in T is the image of S under an embedding from S to T.

Theorem (Leeb)

For $d > 0$ and for ordered trees S, T, there exists an ordered tree V such that for each d-coloring of all copies of S in V there is a copy T' of T in V with

$$\{S' \colon S' \text{ a copy of } S \text{ in } T'\}$$

monochromatic.

$s\colon [n] \to [m]$ is a **rigid surjection** if it is surjective and for each $j \in [n]$,

$$s(j) \leq 1 + \max_{i<j} s(i).$$

Prömel–Voigt: Rigid surjections from $[n]$ to $[m]$ are in a bijective correspondence with m-partitions of n:

$$s \to \mathcal{P}_s = \{s^{-1}(i)\colon i \in [m]\}.$$

Theorem (Graham–Rothschild)

Let $d > 0$. Let k, l be natural numbers. There exists m such that for each d-coloring of all **rigid surjections** *from $[m]$ to $[k]$ there is a rigid surjection $t_0\colon [m] \to [l]$ such that*

$$\{s \circ t_0 \colon s\colon [l] \to [k] \text{ a rigid surjection}\}$$

is monochromatic.

Theorem (Leeb)

Let $d > 0$. Let S, T be ordered trees. There exists an ordered tree V such that for each d-coloring of all **embeddings** *from S to V there exists an embedding $j_0\colon T \to V$ with*

$$\{j_0 \circ i \colon i\colon S \to T \text{ an embedding}\}$$

monochromatic.

On theoretical grounds, one would expect a Ramsey theorem about surjections among trees.

J. Moore: a Ramsey statement equivalent to amenability of Thompson's group

D. Bartošova–A. Kwiatkowska: a Ramsey statement needed for dynamics (computation of the universal minimal flow) of the homeomorphism group of the Lelek fan

Dual Ramsey theorem for trees

Let S be a tree. For $w \in S$, let

$$[w] = \{v \in S \colon v \text{ below } w\}.$$

Let S, T be trees. Let $s \colon T \to S$ and $i \colon S \to T$.
s is **dual to** i if for each $w \in T$

$$s([w]) = i^{-1}([w]).$$

Definition

Let S, T be ordered trees. A function $s\colon T \to S$ is a **rigid surjection** if there is an embedding $i\colon S \to T$ such that s is dual to i.

In this case, s determines i (but i does not determine s).

In the case when S and T are linear orders $[k]$ and $[l]$, the new notion of rigid surjection coincides with the old one.

Theorem (S.)

Let $d > 0$. Let S, T be ordered trees. There exists an ordered tree V such that for each d-coloring of all rigid surjections from V to S there exists a rigid surjection $t_0 \colon V \to T$ with

$$\{s \circ t_0 \colon s \colon T \to S \text{ a rigid surjection}\}$$

monochromatic.

Algebraic notions

The abstract approach reveals the **formal algebraic structure** underlying finite pure Ramsey theorems.

We formulate within this approach an **abstract pigeonhole principle** and an **abstract Ramsey statement**, and prove that the pigeonhole principle implies the Ramsey statement.

Most finite unstructured Ramsey theorems are special instances of the above theorem. The theorem also makes it possible to prove new results.

The following theorems are iterative instances of the general result:

- the classical Ramsey theorem,
- the Hales–Jewett theorem,
- the Graham–Rothschild theorem,
- the versions of these results for partial rigid surjections due to Voigt,
- a new self-dual Ramsey theorem,
- the Milliken Ramsey theorem for finite trees,
- a new common generalization of Deuber's and Jasiński's Ramsey theorems for finite trees,
- Spencer's generalization of the Graham–Rothschild theorem and the Ramsey theorem for affine subspaces (Min Zhao),
- a new dual Ramsey theorem for trees.

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Normed composition spaces

Let $(A, \cdot, \partial, |\cdot|)$ be such that

- $\cdot$ is a partial function from $A \times A$ to A (**multiplication**);
- ∂ is a function from A to A (**truncation**);
- $|\cdot|$ is a function from A to a partially ordered set $(L, \leq)$ (**norm**).

Such a structure with **associative** multiplication is called a **composition space** provided that for $a, b, c \in A$:

(i) if $a \cdot b$ and $a \cdot \partial b$ are defined, then

$$\partial(a \cdot b) = a \cdot \partial b;$$

(ii) $|\partial a| \leq |a|$;

(iii) if $|b| \leq |c|$ and $a \cdot c$ is defined, then so is $a \cdot b$ and $|a \cdot b| \leq |a \cdot c|$.

Dual trees

S, T, V ordered trees, f, g rigid surjections

For $w \in T$, let

$$T^w = \{v \in T\colon v \leq_T w\}$$

For $f\colon T \to S$ and $v \in S$, let

$$f^v = f \restriction T^{i(v)},$$

where i is the injection of f.

Let $w \in T$, $f\colon T^w \to S$ and $g\colon V \to T$. Define

$$g \cdot f = f \circ g^w.$$

A rigid surjection $f\colon T \to S$ is called **sealed** if $f^{-1}(v) = \{w\}$, where v is $\leq_S$-largest in S and w is $\leq_T$-largest in T.

$A =$ all sealed rigid surjections

Multiplication: $\cdot$ defined above.

Truncation: for $f\colon T \to S$ in A, if S consists only of the root, let

$$\partial f = f;$$

otherwise, let v be the second $\leq_S$-largest vertex in S, and let

$$\partial f = f^v.$$

Norm: let

$$L = \text{ all ordered trees}$$

with $S \leq T$ precisely when there is $w \in T$ such that $S = T^w$;
for $f \in A$, let

$$|f| = \mathrm{dom}(f).$$

A with $\cdot$, ∂, $|\cdot|$ defined above is a normed composition space.

Lifting multiplication to sets

Definition

A a normed composition space. Let $\mathcal{F}$ be a family of finite, non-empty subsets of A. Assume we have a partial function from $\mathcal{F} \times \mathcal{F}$ to $\mathcal{F}$:

$$(F, G) \to F \bullet G.$$

We say that $\mathcal{F}$ with this operation is a **family over** A provided that whenever $F \bullet G$ is defined, then $f \cdot g$ is defined for each $f \in F$ and $g \in G$ and

$$F \bullet G = F \cdot G.$$

Dual trees (ctd)

S, T ordered trees
F a non-empty family of $f \in A$ with $f \colon T^w \to S$ for some $w \in T$
Set $r(F) = S$ and $d(F) = T$.

Let $\mathcal{F}$ consist of all F as above.

For $F, G \in \mathcal{F}$, let $F \bullet G$ be defined precisely when $d(G) = r(F)$ and let

$$F \bullet G = F \cdot G.$$

$\mathcal{F}$ is a family of sets over A.

Abstract Ramsey and abstract pigeonhole statements

Ramsey statement

$\mathcal{F}$ a family over a normed composition space

The following condition is our Ramsey statement:

(R) given $d > 0$, for each $P \in \mathcal{F}$,
there is $F \in \mathcal{F}$ such that
$F \bullet P$ is defined and
for every d-coloring of $F \bullet P$ there is $f \in F$ such that $f \cdot P$ is monochromatic.

Dual trees (ctd)

Condition (R) is the dual Ramsey theorem for trees with **sealed** rigid surjections.

Pigeonhole statement

$a \in A$ can be viewed as a partial function from A to A defined on

$$\{x \in A \colon a \cdot x \text{ defined}\}.$$

a is a **restriction of** $b \in A$ if b extends a as a partial function.

For $F \subseteq A$ and $a \in A$, let

$$F^a = \{f \in F \colon a \text{ is a restriction of } f\}.$$

For $P \subseteq A$ and $y \in A$, let

$$P_y = \{x \in P \colon y = \partial x\}.$$

In (R), we color $F \cdot P$ and require making the coloring constant on $f \cdot P$ for some $f \in F$.

Consider the equivalence relation on P given by $\partial x_1 = \partial x_2$, whose equivalence classes are P_y for $y \in \partial P$. Require making the coloring constant on $f \cdot P_y$ for a **fixed** $y \in \partial P$.

Price: make f behave as prescribed by some $a \in A$ on a part of A

$\mathcal{F}$ a family over a normed composition space A.
We consider the following **criterion**:

(P) for $d > 0$, $P \in \mathcal{F}$, and $y \in \partial P$,
there is $F \in \mathcal{F}$ and $a \in A$ such that
$F \bullet P$ is defined, $a \cdot y$ is defined and
for every d-coloring of $F^a \cdot P_y$ there is $f \in F^a$ such that $f \cdot P_y$ is monochromatic.

Main Abstract Theorem

Theorem (S.)

Let $\mathcal{F}$ be a family over a normed composition space. Assume $\mathcal{F}$ fulfills conditions (A), (B), and (C). If for each $F \in \mathcal{F}$ there is $t \in \mathbb{N}$ with $\partial^t F$ having one element, then (P) implies (R).

Dual trees (ctd)

Lemma

$\mathcal{F}$ fulfills conditions (A)–(C).

Lemma

Condition (P) holds for $\mathcal{F}$.

So (R) holds for $\mathcal{F}$ by Main Abstract Theorem.

Lemma

Dual Ramsey theorem for trees can be deduced from the sealed version.

So dual Ramsey theorem for trees **holds**.