Non ergodicity of the incompressible fluid dynamic under vanishing viscosity

Bonn, october 8, 2007

from A.B. Cruzeiro, PM, JFA : http://www.sciencedirect.com

CONTENTS

Basis of vector fields with vanishing divergence

Let T^2 be the 2-dimensional Torus, *let G be the group of volume preserving diffeomorphisms of T^2, let $\mathcal{G}$ be its Lie algebra* which is the space of vector fields with vanishing divergence.The dual group of T^2 is the group Z^2, the lattice of R^2 :

$$< k \ , \ \theta >:= \exp(ik.\theta), \quad k.\theta = k_1\theta_1 + k_2\theta_2.$$

$$f(\theta) = \sum_{k\in Z^2} \hat{f}(k) \ < k \ , \ \theta > .$$

Then f is real if and only if $\hat{f}(-k) = \bar{\hat{f}}(k)$. Introduce $\tilde{Z}^2$ a subset of Z^2 each equivalence defined by $k \simeq k'$ if $k + k' = 0$ has a unique representative in $\tilde{Z}^2$. Then

$$f(\theta) = 2\sum_{k\in \tilde{Z}^2} \Re\hat{f}(k) \ \cos(k.\theta) - \Im\hat{f}(k) \ \sin(k.\theta)$$

Take for $\mathcal{G}$, the constant vector fields and

$$A_k = \frac{1}{|k|}[(k_2 \cos k.\theta)\frac{\partial}{\partial\theta_1} - (k_1 \cos k.\theta)\frac{\partial}{\partial\theta_2}]$$

$$B_k = \frac{1}{|k|}[(k_2 \sin k.\theta)\frac{\partial}{\partial\theta_1} - (k_1 \sin k.\theta)\frac{\partial}{\partial\theta_2}]$$

where $k \in \tilde{Z}^2 - ((0,0)$, where $|k|^2 = k_1^2 + k_2^2$.

Arnold geometrization of hydrodynamic

The kinetic energy is the square of the velocity :

$$\|Z\|^2_{\mathcal{G}} := \int_{T^2} |Z(\theta)|^2 \ \frac{d\theta}{\pi^2} \text{ which}$$

induces on G *a structure of Riemannian manifold :*

fluid motions $\longleftrightarrow$ **geodesics**

Stochastic Atlas

Let $\mathcal{L}$ the differential operator defined on G by

$$2\mathcal{L} = \partial^2_{A_{(1,0)}} + \partial^2_{B_{(1,0)}} + \partial^2_{A_{(0,1)}} + \partial^2_{B_{(0,1)}}$$

The Stratanovitch SDE associated to $\mathcal{L}$ is

$$dg_x(t) = A_{(1,0)}(g_x) \ \circ \ dx^1 + B_{(1,0)}(g_x) \ \circ \ dx^2$$

$$+A_{(0,1)}(g_x) \circ dx^3 + B_{(0,1)}(g_x) \circ dx^4$$

x an R^4-valued Brownian motion, $g_x(0)$ =Identity.
Local Chart $\simeq \ \psi_x : \ [0,1] \mapsto G$ defined by $t \mapsto g_x(t)$.
Stochastic atlas $\mathcal{A}_{\mathcal{L}} \quad \simeq \quad \{\psi_x\}_{x \in X}$.

Christofell symbols

Theorem.
The transfer tensor field is equal to

$$\Gamma^k_{ij} = \frac{1}{2}\big(([e_i,e_j] \mid e_k) - ([e_j,e_k] \mid e_i) + ([e_k,e_i] \mid e_j)\big),$$

The Jacobian : $\forall\ u,\ \eta \in \mathcal{G}$ determine $\zeta \in \mathcal{G}$

$$\exp(\epsilon\zeta)\exp(u) = \exp(u+\epsilon\eta) + o(\epsilon) \quad \epsilon \to 0. \qquad (i)$$

$$\zeta = \int_0^1 \exp(tu)\eta\exp(-tu)\ dt \qquad (ii)$$

$$\zeta = \eta + \frac{1}{2}\mathrm{ad}(u)(\eta) + o(u) \qquad (iii)$$

In the exponential chart the Riemann-Christoffel $\hat{\mathbf{\Gamma}}$:

$$(\hat{\mathbf{\Gamma}}(e_i)(e_j) \mid e_k) = \frac{1}{2}\big[([e_i,e_k]|e_j)\ +\ ([e_j,e_k]|e_i)\big]$$

We compute the right invariant connection in the exponential chart. The identity (iii) leads to

$$\tilde{\Gamma}^k_{ij} = \frac{1}{2}([e_i,e_j] \mid e_k)$$

The difference of the two connections $\mathbf{\Gamma}^k_{ij} = \hat{\Gamma}^k_{ij} - \tilde{\Gamma}^k_{ij}$ gives $\mathbf{\Gamma}^k_{ij}$ $\bullet$

Structural constants

$$\alpha_{k,l} := \frac{1}{|k||l||k+l|}(l \mid (l+k))$$

$$\beta_{k,l} := \frac{1}{|k||l||k-l|}(l \mid (l-k))$$

$$[k,l] := k_1 l_2 - k_2 l_1$$

Theorem.
The constants of structure of $\mathcal{G}$ are given by

$$[A_k, A_l] = \frac{[k,l]}{2|k||l|}(|k+l|B_{k+l} + |k-l|B_{k-l})$$

The Cristoffel symbols are

$$\Gamma_{A_k,A_l} = [k,l](\alpha_{k,l}B_{k+l} + \beta_{k,l}B_{k-l})$$

Corollary
The flows $t \mapsto \exp(tA_k)$ are geodesic
Proof : $\Gamma_{A_k,A_k} = 0$

Piecewise Euler flow approximation of a diffusion
Let Θ be a measure on $\tilde{G}$ infinitesimally invariant under the Euler flow then $\forall\ \Phi$ bounded

$$E\left(\int_{\tilde{G}} \Phi(g_{x,t}\gamma)\ \Theta(d\gamma)\right) = \int_{\tilde{G}} \Phi(\gamma)\ \Theta(d\gamma)$$

We denote by $\mathcal{E}$ the probability space associated to an infinite number of independent random variables $\{\eta_n\}$ taking each with equal probability the values 1 or -1. fix $\epsilon > 0$ and consider the continuous $\tilde{G}$-valued curve $\varphi_{\epsilon,\eta}$ defined for $t \geq 0$ by

$$\varphi_{\epsilon,\eta}(t) = \exp\left((t-k\epsilon^2)\times \eta_k Z_{q(k)}\right)\varphi_{\epsilon,\eta}(k\epsilon^2),$$

where the integer k is determined by the condition $t \in]k\epsilon^2, (k+1)\epsilon^2]$, where $q(k)$ denotes the remainder modulo 4 of the integer k and finally where

$$Z_0 = A_{1,0},\ Z_1 = B_{1,0},\ Z_2 = A_{0,1}, Z_3 = B_{0,1}$$

As φ_ϵ is a sequence of Euler evolution, $\forall\ \eta \in \mathcal{E}\ \forall\ \epsilon > 0$

$$\int_{\tilde{G}} \Phi(\varphi_{\epsilon,\eta}(t)\gamma)\ \Theta(d\gamma) = \int_{\tilde{G}} \Phi(\gamma)\ \Theta(d\gamma) \qquad \bullet$$

Regular representation of diffeomorphism group

Let $\mathcal{U}$ be the unitary group of $L^2(T^2)$, the Hilbert space of complex valued square integrable functions. *The multiplicative unitary subgroup* : let $\mathcal{U}^m$ be the subgroup of the unitary group $\mathcal{U}$ defined as

$$\mathcal{U}^m := \Big\{ U \in \mathcal{U}; \quad U(f_1 f_2) = U(f_1) \times U(f_2), \Big\}$$

Let $\tilde{G}$ be the group of Borel measurable maps of T^2 in itself which preserve the Lebesgue volume measure; define *regular representation* as the map $j : \tilde{G} \mapsto \mathcal{U}^m$ by associating to $g \in G$ the operator

$$U_g(f) = f \circ g, \quad \forall f \in L^2.$$

Theorem

The regular representation is a surjective isomorphism

The representation j induces a morphism j' of Lie algebras; define $\mathcal{A}_k = j'(A_k), \ldots$, then $U_{x,t} := j(g_{x,t})$

$$dU_{x,t} = U_{x,t}(\mathcal{A}_{(1,0)} \circ dx^1 + \mathcal{B}_{(1,0)} \circ dx^2$$

$$+\mathcal{A}_{(0,1)} \circ dx^3 + \mathcal{B}_{(0,1)} \circ dx^4)$$

An SDE on a group of unitary operator

Parametrize U_g by $c_s^q(g) = (U_g(e_s) \mid e_q)$

$$c_s^q(g) := \frac{1}{(2\pi)^2} \int_{T^d} \exp\{-iq.\theta + is.g(\theta)\}\, d\theta \quad \text{then}$$

$$[\mathcal{A}_k c_s^q](g) = \left((D_{A_k^\alpha} U_*)_g(e_s) \,\middle|\, e_q \right)$$

$$= \frac{i}{(2\pi)^d} \int_{T^d} \exp(-i(q.\theta - s.g(\theta)) \times (s.k) A_k(g(\theta))\, d\theta$$

$$\mathcal{A}_k(e_s) = \frac{i}{2}(s.k) \times (e_{s+k} + e_{s-k})$$

$$\mathcal{B}_k(e_s) = \frac{1}{2}(s.k) \times (e_{s+k} - e_{s-k})$$

$$[\mathcal{A}_k]^2 e_s = -\frac{(s.k)^2}{2} e_s - \frac{(s.k)(s.(s+k))}{4} e_{s+2k} + \dots$$

finally

$$([\mathcal{A}_{(0,1)}]^2 + [\mathcal{B}^2_{(0,1)} + [\mathcal{A}_{(1,0)}]^2 + [\mathcal{B}_{(1,0)}]^2) e_s = -|s|^2 \times e_s$$

Transfer energy matrix

Theorem

Consider the coefficients of the matrix $U_{x,t}$

$$c_s^q(x,t) = (U_{x,t} e_s \mid e_q) \quad q, s \in Z^d$$

fixing $q \in Z$*, then the energy functional*

$$\xi_t(s) := E(|c_s^q(x,t)|^2)$$

satisfies the ODE

$$\frac{d\xi_t}{dt} = \mathcal{M}(\xi_t)$$

where $\mathcal{M}$ *is a real symmetric negative matrix which has for asocited quadratic form*

$$(\mathcal{M}(\xi) \mid \xi) = -\frac{s_2^2}{2}\left((\xi_s - \xi_{s+(1,0)})^2 + (\xi_s - \xi_{s-(1,0)})^2\right)$$

$$-\frac{s_1^2}{2}\left((\xi_s - \xi_{s+(0,1)})^2 + (\xi_s - \xi_{s-(0,1)})^2\right)$$

Jump process associted to a Dirichlet form

Rescale the s column of the matrix $\mathcal{M}$ by dividing each term by $-|s|^2$; then we obtain a probability measure carried by the complement of s; making this construction for all s we define a **random walk** $X(n)$ on Z^2. The jumps which can appear are $(1,0),\ -(1,0),\ (0,1),\ -(0,1)$.

The **jump process** is defined as

$$\eta(t) := X(\varphi(t))$$

where the change of clock $\varphi(t)$ is the integer valued function $\varphi(t)$:

$$\sum_{n \leq \varphi(t)} \frac{1}{\mathcal{M}_l^l} \times \Lambda_n \leq t < \sum_{n \leq \varphi(t)+1} \frac{1}{\mathcal{M}_l^l} \times \Lambda_n,$$

and where $\{\Lambda_k\}$ is a sequence of independent exponential times.

The infinitesimal generator of the process η is $\mathcal{M}$

Escape the energy towards high modes

Theorem

$$\textit{For all } j_0 \textit{ fixed,} \ \lim_{t\to\infty} \xi_t(j_0) = 0$$

Proof.

Using the theory of Dirichlet forms we have

$$P_t(f)(k_0) = E_{k_0}(f(\eta(t)))$$

Let ν be the uniform measure defined on Z^2 introduce $\mathcal{Q}(\phi) := \mathcal{M}(\phi) \mid \phi)$, consider the Hilbert space

$$\|\phi\|_{\mathcal{D}}^2 := \mathcal{Q}(\phi) + \|\phi\|_{L_\nu^2}^2$$

Let $\mathcal{D}$ be the associated Hilbert space constructed by completing denote $\mathcal{N}$ the operator $\mathcal{D} \to L_\nu^2$ define by the closure of $\mathcal{M}$ in $\mathcal{H}$. Then $\mathcal{N}$ is a selfadjoint operator and we have

$$\|f\|_{\mathcal{D}}^2 := (\mathcal{N}f \mid f)_{L_\nu^2} + \|\phi\|_{L_\nu^2}^2$$

Spectral Theorem

$$\mathcal{N} = \int_{-\infty}^{0} \lambda \, d\Pi(\lambda)$$

where $\Pi(\lambda)$ is an orthogonal projection operator in the Hilbert space L^2_ν, the map $\lambda \mapsto \text{Image}(\Pi(\lambda))$ being an increasing function with values in the closed subspaces of L^2_ν; then define

$$P_t f = \int_{-\infty}^{0} \exp(t\lambda) d(\Pi(\lambda) f, \quad \forall \; f \in \mathcal{H}.$$

$$\|P_t f\|^2_{l^2} = \int_{-\infty}^{0} \exp(t\lambda) d(\|\Pi(\lambda) f\|^2)$$

This integral does not converge to 0 if and only if the measure $d(\|\Pi(\lambda) f\|^2)$ has a Dirac mass at the origin, wich means that there exists

$$\psi \in L^2_\nu, \; \psi \neq 0, \; \text{such that } \mathcal{Q}(\psi) = 0$$

Absurd $\longrightarrow \lim_{t \to \infty} \|P_t f\|_{L^2_\nu} = 0 \quad \bullet$