On a Liouville type theorem for harmonic maps to convex spaces via Markov chains

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1 History

Liouville Th.: (up/low) bdd harmonic func on $\mathbb{R}^d$ is const.

Harmonic map: (M, g), (N, h): compl C^∞ Riem. mfd,

$C^\infty(M, N)$: smooth maps from M to N.

$E(u) := \int_M e(u)(x)\mu_g(dx)$: energy of $u \in C^\infty(M, N)$

$$e(u)(x) := \frac{1}{2}g^{ij}(h_{\alpha\beta}(u(x))\frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\beta}{\partial x^j} \text{ for } u = (u^\alpha(x^i))$$

$u \in C^\infty(M, N)$ is called harmonic if it is a critical point of E, that is, for any variation u_t of $u_0 = u$

$$\frac{d}{dt}E(u_t)\Big|_{t=0} = 0.$$

Eells-Sampson (1964): M, N: compact, (N, h): NPC space, $\exists$ harmonic map from M to N in each free-homotopy class.

Eells-Sampson (1964): M, N: same as above. If $\mathrm{Ric}_M > 0$, then any harmonic map $u : M \to N$ is a constant.

Schoen-Yau (1976): M, N: complete, N: NPC, if $\mathrm{Ric}_M \geq 0$, any harmonic map $u : M \to N$ of finite energy is a constant.

Cheng (1980): M, N: complete, N: Hadamard, if $\mathrm{Ric}_M \geq 0$, any harmonic map $u : M \to N$ with sublinear growth is a constant.

Hildebrandt-Jost-Widman (1981): M: simple or compact, any harmonic map form M to a regular geodesic ball is a constant map.

Choi (1982): M: complete, $\text{Ric}_M \geq 0$. Any harmonic map form M to a regular geodesic ball is a constant map.

Yau (1975) **Cheng-Yau** (1975): M: complete, Ric_M is bdd below. Gradient estimate of positive harmonic function on M. Elliptic Harnack Ineq. holds for M, hence strong Liouville property holds.

Kendall (1988): M: complete and if Liouville property holds for bdd harmnic function on M, then any bdd harmonic map from M to a regular geodesic ball is a constant map. (Stochastic Proof).

Stafford (1990): Stochastic proof for Cheng (1980).

Tam (1995): M: quasi-isometric to compl. R-mfd with Ric>0. Any harmonic map from M to Hadamard N with

$$d_N(u(x), u(x_0)) = o(d_M(x, x_0)^\alpha), \; x \in M, \; \alpha \in]0,1]$$

is a constant map.

Sung-Tam-Wang (1996): M: non-compact R-mfd. If a bdd harmonic funcion asymptotically constant on each non-parabolic end of M, then any bdd harmonic map is a constant map on each non-parabolic end of M.

Cheng-Tam-Wan (1996): M: complete, if (LP) holds for functions on M, then any harmonic map to Hadamard manifolds of finite energy is a constant map.

Atsuji (2006): L: a sym. elliptic op. on M. If (LP) holds for L-harmonic function on M, then any harmonic map to Hadamard manifolds of finite energy w.r.t. E_L is a constant map. (Stochastic Proof)

2 Target Space

(Y, d) : complete separable geodesic space

Def 2.1 (Admissible function) $\Phi : Y \times Y \to [0, \infty[$ is said to be *admissible* if

(1) $\Phi(x, y) = 0$ iff $x = y$.

(2) $y \mapsto \Phi(x, y)$ is continuous for $x \in Y$, $x \mapsto \Phi(x, y)$ is u.s.c. for $y \in Y$,

(3) $\exists$ u.s.c. $\psi : [0, \infty[\to [0, \infty[$ s.t. $\psi > 0$ on $]0, \infty[$, $\psi(0) = 0$ and $\Phi(x, y) \le \psi(d(x, y))$ for $x, y \in Y$.

(Y, d, Φ) is said to be an *admissible space* if Φ is admissible.

$\mathcal{P}(Y)$: Borel probability measures on Y. For an admissible function Φ, we set

$$\mathcal{P}^{\Phi}(Y) := \left\{ \mu \in \mathcal{P}(Y) \;\middle|\; \int_{Y} \Phi(w,z)\mu(dw) < \infty \text{ for all } z \in Y \right\}.$$

When $\Phi = d^p$, we write $\mathcal{P}^{\Phi}(Y)$ instead of $\mathcal{P}^{p}(Y)$.

Def 2.2 (Φ-barycenter) Let (Y, d, Φ) be an admissible space. For $\mu \in \mathcal{P}^{\Phi}(Y)$, a point $b(\mu) \in Y$ is called the Φ-*barycenter* of μ iff for each $z \in Y$,

$$\Phi(b(\mu), z) \leq \int_{Y} \Phi(w, z)\mu(dw)(< \infty). \tag{1}$$

Rem 2.1 The Φ-barycenter of $\mu \in \mathcal{P}^{\Phi}(Y)$ is an analogy of the barycenter defined by Doss (1949) or Herer (1986), which is NOT UNIQUE in general.

Denote by $B^{\Phi}(\mu)$ the family of Φ-barycenter of μ.

There are many examples of admissible spaces (Y, d, Φ) with Φ-barycenters:

Ex 2.1 (CAT(0)-Space) A complete separable metric space (Y, d) is called the *CAT(0)-space* (*Hadamard space, or global NPC space*) if for any pair of points $\gamma_0, \gamma_1 \in Y$ and any $t \in [0, 1]$ there exists a point $\gamma_t \in Y$ such that for any $z \in Y$

$$d^2(z, \gamma_t) \leq (1-t)d^2(z, \gamma_0) + td^2(z, \gamma_1) - t(1-t)d^2(\gamma_0, \gamma_1). \tag{2}$$

By definition, $\gamma := (\gamma_t)_{t \in [0,1]}$ is the mini. geodesic joining γ_0 and γ_1 and Y is simply connected. Hadamard mfd, Euclidean building (Metric tree), Hilbert sp. are typical examples of CAT(0)-spaces. GH-limits of CAT(0), ℓ^2-type products of CAT(0), L^2-maps to CAT(0) is also CAT(0).

The following is well-known:

(1) The distance function $d : Y \times Y \longrightarrow [0, \infty[$ is convex.

(2) For $\mu \in \mathcal{P}^1(Y)$, the barycenter $b(\mu)$ of μ is defined as the unique minimizer of

$$Y \ni x \mapsto \int_Y (d^2(x, y) - d^2(z, y))\mu(dy),$$

which is independent of $z \in Y$.

(3) For any l.s.c. convex function f on Y, Jensen's inequality holds:

$$f(b(\mu)) \leq \int_Y f(x)\mu(dx).$$

In particular, $d(b(\mu), x_0) \leq \int_Y d(x, x_0)\mu(dx)$holds.

Ex **2.2 (*k*-Convex Space)** **A complete separable metric space (Y, d) is called the *k-convex space* or 2-*uniformly convex space* if (Y, d) is a geodesic space and for any three points $x, y, z \in Y$, any geodesic $\gamma := (\gamma_t)_{t\in[0,1]}$ in Y with $\gamma_0 = x$, $\gamma_1 = y$, and all $t \in [0, 1]$,**

$$d^2(z, \gamma_t) \leq (1-t)d^2(z, x) + td^2(z, y) - \frac{k}{2}t(1-t)d^2(x, y). \tag{3}$$

By definition, putting $z = \gamma_t$, we see $k \in]0, 2]$. The inequality (3) yields the (strict) convexity of $Y \ni x \mapsto d^2(z, x)$ for a fixed $z \in Y$.

Examples of k-convex spaces:

(1) **Any closed convex subset of a k-convex space is again k-convex.**

(2) **Every CAT(0)-space is a 2-convex space.**

(3) **Any CAT(1)-space Y with diam$(Y) \leq \frac{\pi}{2} - \varepsilon$, $\varepsilon \in]0, \frac{\pi}{2}[$ is a $\{(\pi - 2\varepsilon)\tan\varepsilon\}$-convex space (by S. Ohta (2007)).**

(4) **Any Banach space L^p with $p \in]1, 2]$ over a measurable space is a $2(p-1)$-convex space (by S. Ohta (2007)).**

Ohta (2007) proved that any two points in a k-convex space can be connected by a unique minimal geodesic and contractible.

We can define the barycenter of $\mu \in \mathcal{P}^1(Y)$ for k-convex space Y. It is unclear that Jensen's ineq holds or not in the framework of k-convex spaces.

Consider the unique minimizer $\bar{b}(\mu)$ of

$$C(\mathrm{supp}[\mu]) \ni x \mapsto \int_Y (d^2(x,y) - d^2(z,y))\mu(dy)$$

and call it Pure Barycenter. Then Jensen's ineq holds for the pure barycenter. Here $C(\mathrm{supp}[\mu])$ is the closed convex hull of the support of μ.

Rem 2.2 If (Y, d) is a CAT(1)-space with $\mathrm{diam}(Y) < \pi/2$, then the pure barycenter of $\mu \in \mathcal{P}^1(Y)$ coincides with the barycneter. So Jensen' ineq holds for the barycenter.

Ex 2.3 (Regular Geodesic Ball in R-Mfd)

(M, g): m-dim C^∞ R-mfd. d : distance derived from g.

$Y := \{x \in M \mid d(o, x) \leq r\}$ is called the regular geodesic ball if $Y \cap \mathrm{Cut}_o = \emptyset$ and the upper bound of the sectional curvature $\kappa(= \kappa(r))$ satisfies $0 \leq \kappa < (\frac{\pi}{2r})^2$. When $\kappa > 0$ we put

$$\Phi(w, z) := \frac{1}{\kappa} \cdot \frac{1 - \cos(\sqrt{\kappa} d(w, z))}{\cos(\sqrt{\kappa} d(w, o))}.$$

Then $w \mapsto \Phi(w, z)$ is convex for each $z \in Y$ and $\Phi(w, z) \leq \frac{d^2(w,z)}{2 \cos \sqrt{\kappa} r}$ for $w, z \in Y$.

The barycenter of $\mu \in \mathcal{P}(Y)$ is defined as the local minimizer of

$$Y \ni z \mapsto \int_Y d^2(z, w)\mu(dw)$$

This is also called the *Karcher Mean* of μ. Kendall(1990) proved that Karcher mean $b(\mu)$ uniquely exists and Jensen's ineq holds :

For any bounded convex function φ

$$\varphi(b(\mu)) \leq \int_Y \varphi(w)\mu(dw). \tag{4}$$

Hence, for each $z \in Y$, we have $\Phi(b(\mu), z) \leq \int_Y \Phi(w, z)\mu(dw)$.

Ex 2.4 (Banach Space) **Let (Y, d) be a separable Banach space, where d is given through the norm on Y. In this case, Φ can be taken to be the distance d and Φ-barycenter can be given by the barycenter $b(\mu)$ for $\mu \in \mathcal{P}^1(Y)$ defined as the Bochner integral $\int_Y x\mu(dx)$. By the property of Bochner integral, we have**

$$\|b(\mu) - z\| \leq \int_Y \|x - z\|\mu(dx).$$

so that $b(\mu)$ is a Φ-barycenter. Then (Y, d, Φ) is an admissible space admitting Φ-barycenters.

Ex **2.5 (Non-uniqueness of Φ-barycenter)**

We set $(Y, d) = (\mathbb{R}^2, d_{\ell^p})$, $p \in]1, 2[$ with

$$d_{\ell^p}(x, y) := \|x - y\|_p = (|x_1 - x_2|^p + |y_1 - y_2|^p)^{1/p},$$

$x = (x_1, x_2), y = (y_1, y_2)$. Set $\Phi = d_{\ell^p}$. Then (Y, d, Φ) is admissible. Take $0 = (0, 0)$, $e_1 := (1, 0)$, $e_2 = (0, 1) \in \mathbb{R}^2$ and $\mu := \frac{1}{3}\delta_{e_1} + \frac{1}{3}\delta_0 + \frac{1}{3}\delta_{e_2}$. We see that the Bochner integral of x by μ is given by

$$\int_{\mathbb{R}^2} x\mu(dx) = \left(\frac{1}{3}, \frac{1}{3}\right).$$

Note that any line segment between two points in $\mathbb{R}^2$ is the minimal geodesic in $(\mathbb{R}^2, d_{\ell^p})$ joining them.

For $x = (x_1, x_2) \in \mathbb{R}^2$, set

$$f_p(x_1, x_2) := 3 \int_{\mathbb{R}^2} d^2_{\ell^p}(x, y)\mu(dy) = (|1 - x_1|^p + |x_2|^p)^{\frac{2}{p}} + (|x_1|^p + |x_2|^p)^{\frac{2}{p}} + (|x_1|^p + |1 - x_2|^p)^{\frac{2}{p}} .$$

Then $b(\mu) \in \mathbb{R}^2$ (resp. $\bar{b}(\mu) \in \triangle 0 e_1 e_2$) is the unique minimizer of $\mathbb{R}^2 \ni (x_1, x_2) \mapsto f_p(x_1, x_2)$ (resp. $\triangle 0 e_1 e_2 \ni (x_1, x_2) \mapsto f_p(x_1, x_2)$).

$$f_p\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{2(2^p + 1)^{2/p} + 2^{2/p}}{9} \longrightarrow \frac{22}{9} > 2 = f_p(0, 0) \geq f_p(\bar{b}(\mu)) \geq f_p(b(\mu))$$

as $p \longrightarrow 1$, we have $b(\mu) \neq (1/3, 1/3)$ and $\bar{b}(\mu) \neq (1/3, 1/3)$ for some $p \in]1, 2[$.

3 Domain

$(X, \mathcal{X})$: measurable space.

$P(x, A)$: Markov kernel on $(X, \mathcal{X})$, i.e.

(1) $^\forall x \in X$, $P(x, \cdot)$ is a probability measure over $(X, \mathcal{X})$,

(2) $^\forall A \in \mathcal{X}$, $x \mapsto P(x, A)$ is $\mathcal{X}$-measurable.

$^\exists$M$= (\Omega, X_n, \mathcal{F}_\infty, \mathbb{P}_x)_{x \in X}$: conservative Markov chain s.t.

$P(x, dy) = \mathbb{P}_x(X_1 \in dy)$: transition kernel of M.

For an $\mathcal{X}$-measurable f on X,

$$Pf(x) := \int_X f(y) P(x, dy) = \mathbb{E}_x[f(X_1)].$$

Def 3.1 (P-Harmonic Map) Let (Y, d, Φ) be an admissible space and $u : X \to Y$, $\mathcal{X}/\mathcal{B}(Y)$-measurable map s.t. $u_*P(x,\cdot) \in \mathcal{P}^\Phi(Y)$. We set

$$Pu(x) := b(u_*P(x,\cdot)).$$

Here $u_*P(x,\cdot)$ is a Borel measure defined by $u_*P(x,\cdot)(A) := P(x, u^{-1}(A))$, $A \in \mathcal{B}(Y)$. We choose $b(u_*P(x,\cdot))$ from $B^\Phi(u_*P(x,\cdot))$. An $\mathcal{X}$-measurable $f : X \to \mathbb{R}$ is said to be *P-subharmonic* if $f \le Pf$ on X. For $\mathcal{X}/\mathcal{B}(Y)$-measurable map $u : X \to Y$ is called *P-harmonic* if $u = Pu$ on X.

4 Result

Thm 4.1 (Liouville Property) Let (Y, d, Φ) be an admissible space. We assume one of the following:

(1) (Y, d) is proper, that is, any bdd closed subset is compact.

(2) (Y, d) is a separable Banach space with $\Phi = d$ and $b(\mu) = \int_Y x\mu(dx)$ for $\mu \in \mathcal{P}^1(Y)$.

If any bdd P-harmonic func. on X is always constant, then the same property holds for any bdd P-harmonic map.

Rem 4.1 (Open Problem) **The conclusion of Thm 4.1 holds or not when (Y, d) is a non-proper CAT(0)-space except the case of Hilbert space?**

Lem 4.1 Let (Y, d, Φ) be an admissible space. If an $\mathcal{X}/\mathcal{B}(Y)$-measurable map $u : X \longrightarrow Y$ satisfying $u_*P(x, \cdot) \in \mathcal{P}^{\Phi}(Y)$ is a bounded P-harmonic map, then $x \mapsto \Phi(u(x), y_0)$ is a bounded P-subharmonic function for each $y_0 \in Y$.

Proof : From the definition of Φ-barycenter of $u_*P(x, \cdot)$,

$$\Phi(u(x), y_0) = \Phi(Pu(x), y_0) \leq P\Phi(u(\cdot), y_0) < \infty.$$

$\square$

Sketch of the proof of Thm 4.1:

Let $u : X \to Y$ be a bounded P-harmonic function. we may assume $\mathrm{Im}(u) \subset B_R(o)$ for some $o \in Y$.

Put $M_y := \sup_{x\in X} \Phi(u(x), y) < \infty$.

Then $h_y(x) := M_y - \Phi(u(x), y)$ is P-superharmonic.

(Riesz Decomposition for P-Superharmonic Functions):

$$h_y = h_{\infty,y} + Gg_y,$$

$h_{\infty,y}(x) := \lim_{n\to\infty} P^n h_y(x)$: P-harmonic part,

Gg_y with $g_y := (I - P)h_y \geq 0$: Potential part,

$\lim_{n\to\infty} P^n Gg_y = 0$. Here $G := \sum_{n=0}^{\infty} P^n$.

By assumption $h_{\infty,y} \equiv c_y$. Putting $d_y := M_y - c_y$, we have $d_y - \Phi(u(x), y) = Gg_y(x)$, $x \in X$, hence $\Phi(u(x), y) \leq d_y$, $x \in X$.

Let $\{y_i\}$ be a countable dense subset of Y and put $g_i := g_{y_i}$. For any $z \in X$, $r > 0$ and $N \in \mathbb{N}$, the following (5) holds for some $^\exists n \in \mathbb{N}$:

$$P^n\left(z, \bigcap_{i=1}^{N}\left\{x \in X \,\middle|\, Gg_i(x) \leq r\right\}\right) > 0. \tag{5}$$

Indeed, if (5) does not hold, then there exist $z \in X$, $r > 0$ and $N \in \mathbb{N}$ s.t.

$$G(g_1 + g_2 + \cdots + g_N) > r \quad P^n(z, \cdot)\text{-a.s. on } X \text{ for all } n \in \mathbb{N}.$$

Integrate the both sides by $P^n(z, dy)$, the left hand side is greater than r on X, which contradicts $\lim_{n\to\infty} P^n G(g_1 + g_2 + \cdots + g_N)(z) = 0$.

From (5)

$$\bigcap_{i=1}^{N} \left\{ x \in X \;\middle|\; Gg_i(x) \le r \right\} \ne \emptyset,$$

in particular,

$$\bigcap_{i=1}^{N} \left\{ y \in \bar{B}_R(o) \;\middle|\; d_{y_i} - \Phi(y, y_i) \le r \right\} \ne \emptyset.$$

When (Y, d) is proper, $\bar{B}_R(o)$ is compact. Hence countable intersection of closed decreasing non-empty subsets of $\bar{B}_R(o)$ is non-empty. Therefore

$$\bigcap_{n=1}^{\infty} \bigcap_{i=1}^{\infty} \left\{ y \in \bar{B}_R(o) \;\middle|\; d_{y_i} - \Phi(y, y_i) \le 1/n \right\} \ne \emptyset.$$

So $^{\exists} y_0 \in \bar{B}_R(o)$ s.t.

$$d_{y_i} - \Phi(y_0, y_i) \le 1/n \quad {}^{\forall} i, n \in \mathbb{N},$$

in particular, $d_{y_i} \leq \Phi(y_0, y_i) \quad \forall i \in \mathbb{N}$. Thus

$$\Phi(u(x), y_i) \leq \Phi(y_0, y_i), \forall x \in X, i \in \mathbb{N},$$

hence $u \equiv y_0$.

When Y is a separable Banach space.

$u : X \to Y$: bounded P-harmonic map. For $\ell \in Y^*$, $x \mapsto {}_{Y^*}\langle \ell, u(x) \rangle_Y$ is a P-harmonic function in view of the property of Bochner integral. By assumption, we have

$$ {}_{Y^*}\langle \ell, u(x) \rangle_Y \equiv c_\ell. $$

For any $x, y \in X$,

$$ \| u(x) - u(y) \|_Y = \sup_{\ell \in Y^*, \|\ell\|_{Y^*} \le 1} {}_{Y^*}\langle \ell, u(x) - u(y) \rangle_Y = 0. \quad (6) $$

We obtain $u \equiv x_0$ for some $x_0 \in Y$. □

Thank you for

your attention!

Harmonic map between singular objects by geometers:

Gromov-Schoen (1992): Harmonic map from domains in R-mfd to Riemannan simplicial complex which is a loc cpt CAT(0)-space embedded in $\mathbb{R}^d$.

Korevaar-Schoen (1993): Harmonic map from domains in R-mfd to CAT(0)-space.

Jost (1995)(1997): Harmonic maps between metric spaces.

Kotani-Sunada (2001): Application of harmonic maps to crystal lattice.

Izeki-Nayatani (2006): Harmonic maps from simplicial complex to CAT(0)-space.

Harmonic map between singular objects by probabilists:

Picard (2001): Dirichlet Problem for harmonic maps from the space admitting sym. diffusions to Hadamard mfds.

Picard (2005): Stochastic calculus over Tree.

Sturm (2001/2002): Harmonic Maps from the space admitting Markov chains to CAT(0)-space and CAT(0)-valued (non-linear) discrete time martingales.

Christiansen-Sturm (2007?): CAT(0)-valued (non-linear) continuous time martingales.

Christiansen (2006): Stochastic calculus over Riemannian polyhedra (Ph.D. Thesis)

Harmonic map between singular objects by potential theorists:

Eells-Fuglede (2001): Harmonic maps between Riemannian polyhedra.

Fuglede (2003): Hölder continuity of harmonic maps from Riemannian polyhedra to spaces of upper bdd curvature.

Fuglede (2003): Harmonic maps of finite energy from Riemannian polyhedra.

Fuglede (2005): Dirichlet problem for harmonic maps from Riemannian polyhedra to spaces of upper bdd curvature.

Fuglede (2005): Dirichlet problems for harmonic maps from regular domains.