Intrinsic coupling under lower Ricci curvature bounds

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§1 Introduction

M : m-dim. complete Riemannian manifold

$\mathbf{P}_x$: the Wiener measure on $C([0, \infty) \to M)$ for the Brownian motion starting at $x \in M$

Coupling of Brownian motion:

$\boldsymbol{W}_t = (W^{(1)}_t, W^{(2)}_t)$: $M \times M$-valued stochastic process defined on $(\Omega, \mathcal{F}, \mathrm{P})$,

$$\mathrm{P} \circ (W^{(1)})^{-1} = \mathrm{P}_x,$$

$$\mathrm{P} \circ (W^{(2)})^{-1} = \mathrm{P}_y.$$

Examples on $\mathbf{R}^m$

- (X_t, Y_t): independent Brownian motions.
- $(\bar{X}_t, \bar{Y}_t)$: synchronous coupling, given by
$$\bar{Y}_t := \bar{X}_t - x + y.$$
- $(\hat{X}_t, \hat{Y}_t)$: mirror coupling, given by
$$\hat{Y}_t := \begin{cases} R\hat{X}_t & \text{if } t < \tau, \\ \hat{X}_t & \text{if } t \geq \tau. \end{cases}$$

R: reflection w.r.t. a hyperplane, $Rx = y$,
$\tau := \inf\{t > 0 \mid \hat{X}_t = \hat{Y}_t\}$: coupling time.

Kendall '86, Cranston '91: construct "infinitesimally mirror" coupling $(\hat{X}_t, \hat{Y}_t)$ on M by using SDE method.

Under $\mathrm{Ric}_M \geq -(m-1)K$,

$$\mathrm{P}[\tau = \infty] \leq \frac{(m-1)\sqrt{K}}{2} d(x,y).$$

$\Rightarrow \forall u$: bounded harmonic,

$$|u(x) - u(y)| \leq (m-1)\sqrt{K} \|u\|_\infty d(x,y).$$

von Renesse '03: constructs

"infinitesimally mirror" coupling $(\hat{X}_t, \hat{Y}_t)$ on M

by random walk approximation.

Derives the same estimate under $\mathbf{Sect}_M \geq -K$.

This talk:

Random walk approximation

under $\mathbf{Ric}_M \geq -(m-1)K$.

Assumption: $K \geq 0$ (for simplicity).

§2 Geodesic random walks and their couplings

- Fix $\Phi \in \Gamma(\mathcal{O}(M))$ m'ble.
- $\{\xi_n\}_{n\in\mathbb{N}}$: i.i.d.,

  uniform distribution on unit ball in $\mathbb{R}^m$.
- $\{X_n^\varepsilon\}_{n\in\mathbb{N}}$: geodesic random walk given by

$$\begin{cases} X_0^\varepsilon := x, \\ X_{n+1}^\varepsilon := \exp_{X_n^\varepsilon}(\varepsilon\sqrt{m+2}\,\Phi_{X_n^\varepsilon}\xi_n). \end{cases}$$

- $\{X_t^\varepsilon\}_{t\geq 0}$: piecewise geodesic interpolation of $\{X_n^\varepsilon\}_{n\in\mathbb{N}}$.

† $$\frac{1}{\varepsilon^2}\left\{\mathbf{E}[f(\exp_z(\varepsilon\sqrt{m+2}\,\Phi_z\xi_1))] - f(z)\right\} \underset{\varepsilon\to 0}{\longrightarrow} \frac{1}{2}\Delta f(z)$$

† $\{X^{\varepsilon}_{\varepsilon^{-2}t}\}_{t\geq 0}$ converges in law to the Brownian motion starting at x.

Construction of coupled geodesic random walk

Idea

Reflect the infinitesimal motion $dW_t^{(1)}$ and $dW_t^{(2)}$ at the tangent space $T_{W_t^{(1)}}M$ by using the parallel transport $/\!/_t \ : \ T_{W_t^{(2)}}M \longrightarrow T_{W_t^{(1)}}M$ along a minimal geodesic joining $W_t^{(1)}$ and $W_t^{(2)}$.

Lemma(vR.'03)

$\exists$ a map

$$(x, y) \in M \times M$$

$$\downarrow \qquad : \text{ m'ble,}$$

$$\gamma_{xy} \in C([0,1] \to M)$$

where γ_{xy} is a minimal geodesic joining x and y

satisfying $\gamma_{yx} = \gamma_{xy}^{-1}$.

- $\tilde{m}_{xy} \ : \ T_x M \to T_x M$: reflection w.r.t. the subspace perpendicular to $\dot{\gamma}_{xy}(0)$.

- $m_{xy} := /\!/_{\gamma_{xy}} \circ \tilde{m}_{xy}$.

Define $\Phi' \ : \ M \times M \to \mathcal{O}(M)$ by

$$\begin{cases} \Phi'_{(x,y)} = m_{xy} \Phi_x & \text{if } x \neq y \\ \Phi'_{(x,x)} = \Phi_x \end{cases}$$

† $\Phi'_{(x,y)} \in \mathcal{O}_y(M)$.

$Z_n^\varepsilon = (X_n^\varepsilon, Y_n^\varepsilon)$ coupled geodesic random walk defined by

$$\begin{cases} Z_0^\varepsilon &:= (x, y), \\ X_{n+1}^\varepsilon &:= \exp_{X_n^\varepsilon}\left(\varepsilon\sqrt{m+2}\,\Phi_{X_n^\varepsilon}\xi_n\right), \\ Y_{n+1}^\varepsilon &:= \exp_{Y_n^\varepsilon}\left(\varepsilon\sqrt{m+2}\,\Phi'_{Z_n^\varepsilon}\xi_n\right). \end{cases}$$

$\{Z_t^\varepsilon\}_{t\ge 0}$: piecewise geodesic interpolation of Z_n^ε.

⋆ $\{Z^\varepsilon_{\varepsilon^{-2}t}\}_{t\ge 0}$: tight

$\Rightarrow$ $\exists$ a (subsequential) limit $Z_\cdot$ as $\varepsilon \to 0$.

§3 Estimate of coupling time

2nd variational formula & Index lemma

$\Downarrow$

γ: a minimal geodesic joining x and y

For each $\zeta, \eta \in T_x M$ with $|\zeta|, |\eta| \leq 1$,

$$d(\exp_x(t\zeta), \exp_y(t /\!\!/_\gamma \eta)) \leq d(x,y) + t\langle \eta - \zeta, \dot{\gamma}\rangle + \frac{t^2}{2} I_\gamma(V^\perp_{\zeta,\eta}, V^\perp_{\zeta,\eta}) + o(t^2)$$

- $I_\gamma(V,V) := \int_\gamma |\nabla_\gamma V|^2 - \langle \mathcal{R}(V,\dot\gamma)\dot\gamma, V\rangle$

- $V^{\perp}_{\zeta,\eta}$: a vector field along γ with the boundary condition $\zeta^\perp$ and $/\!/_\gamma \eta^\perp$ $(\zeta^\perp, \eta^\perp \perp \dot\gamma)$.

  ⋆ $V^{\perp}_{\zeta,\eta}$ is <u>the Jacobi field</u>

  <u>IF M has a constant sectional curvature K</u>.

- $o(t^2)$ is uniformly small as long as (x,y) moves in a compact region.

$$d(Z_{n+1}^{\varepsilon}) \leq d(Z_n^{\varepsilon}) + 2\varepsilon \lambda_n + \frac{\varepsilon^2}{2}\Lambda_n + o(\varepsilon^2)$$

$$\Downarrow$$

$$d(Z_{\varepsilon^{-2}t}^{\varepsilon}) \leq d(x,y) + 2\varepsilon \sum_{i=1}^{\lfloor \varepsilon^{-2}t \rfloor} \lambda_i + \frac{\varepsilon^2}{2} \sum_{i=1}^{\lfloor \varepsilon^{-2}t \rfloor} \Lambda_i + \text{(remainder)}$$

⋆ $\{\lambda_i\}_i$: i.i.d., $\mathbf{E}[\lambda_1] = 0$, $\mathrm{Var}(\lambda_1) = 1$.

$$\tau_\delta(Z^\varepsilon) := \inf\{t > 0 \mid d(Z_t^\varepsilon) \leq \delta\}.$$

Goal: estimate for $\mathrm{P}[\tau_0(Z) = \infty]$.

Outline of the proof

(i) Localize the range of RW in a ball of radius R.

(ii) Control the sum of Λ_i.

(iii) Give a estimate for $\mathrm{P}[\tau_\delta(Z^\varepsilon) > T]$.

(iv) $\varepsilon \to 0$, $R \to \infty$, $\delta \to 0$, $T \to \infty$.

How do we control the sum of Λ_i?

vR.'03:

$\mathrm{Sect}_M \geq -K \Rightarrow \Lambda_i \leq 2\sqrt{K}|\lambda_i^{\perp}|^2.$

($\lambda_i^{\perp}$: i.i.d., $\{\lambda_i^{\perp}\} \coprod \{\lambda_n\}$, $\mathbf{E}[|\lambda_i^{\perp}|^2] = m - 1$)

$\Rightarrow$ the CLT for λ_i's and the LLN for $\lambda_i^{\perp}$'s work independently!

K.:

$\mathrm{Ric}_M \geq -(m-1)K$

$$\Rightarrow \mathbf{E}[\Lambda_i] \leq 2(m-1)\sqrt{K}.$$

More precisely, $\bar{\Lambda}_i \leq 2(m-1)\sqrt{K}$.

$(\bar{\Lambda}_n := \mathrm{E}[\Lambda_n \mid \mathcal{F}_{n-1}], \mathcal{F}_n := \sigma(\xi_1, \ldots, \xi_n))$

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σ_R: exit time of Z_n^ε from a ball of radius R.

$$E := \left\{ \sup_{0 \leq t \leq T \wedge \sigma_R} \left(\frac{\varepsilon^2}{2} \sum_{i=1}^{\lfloor \varepsilon^{-2} t \rfloor} (\Lambda_i - \bar{\Lambda}_i) \right) \leq \frac{\delta}{4} \right\}$$

Proposition

$\lim_{\varepsilon \downarrow 0} \mathrm{P}[E] = 1$ for any $\delta > 0$.

On E, for $t \leq T \wedge \sigma_R$,

$$d(Z^{\varepsilon}_{\varepsilon^{-2}t}) \leq d(x,y) + 2\varepsilon \sum_{i=1}^{\lfloor \varepsilon^{-2}t \rfloor} \lambda_i + (m-1)\sqrt{K}t + \frac{\delta}{2}$$

$$=: r^{\varepsilon}(t) + \frac{\delta}{2}.$$

$$A := \left\{ \inf_{0 \leq t \leq T \wedge \sigma_R} r^{\varepsilon}(t) \leq \frac{\delta}{2} \right\}.$$

$$\mathbf{P}[\tau_\delta(Z^\varepsilon) > T] \approx \mathbf{P}[\{\tau_\delta(Z^\varepsilon) > T\} \cap E] \le \mathbf{P}[A \cap E] \approx \mathbf{P}[A].$$

$$\Downarrow$$

$$\mathbf{P}[\tau_0(Z) > T] \le \mathbf{P}\left[\inf_{0 \le t \le T} r(t) > 0\right].$$

$$r(t) := d(x,y) + 2\beta(t) + (m-1)\sqrt{K}t,$$

β: 1-dim. standard BM.

$$\therefore \mathbf{P}[\tau_0(Z) = \infty] \le \frac{(m-1)\sqrt{K}}{2} d(x,y).$$

§4 Further remarks

Obstruction to generalization

to singular spaces with lower curvature bounds:

implicit use of local upper sectional curvature bounds.

- variational formula (local injectivity radius),
- control of Λ_i and $\bar{\Lambda}_i$.

A generalization

M: an Alexandrov space satisfying the following:

$\exists \{x_i\}_{i=1}^N$ s.t.

- $M \setminus \{x_i\}_{i=1}^N$: smooth Riem. mfd, $\mathrm{Ric} \geq -(m-1)K$ on $M \setminus \{x_i\}_{i=1}^N$,
- $\mathrm{diam}(\Sigma_{x_i}(M)) < \pi$,
- $\mathrm{Cap}(\{x_i\}) = 0$.

$$\Rightarrow \mathrm{P}[\tau_0(Z) = \infty] \leq \frac{(m-1)\sqrt{K}}{2} d(x, y)$$

for $x, y \in M \setminus \{x_i\}_{i=1}^N$.