Integrated Harnack Inequalities

Joint with Masha Gordina

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Example; $G = \mathbb{R}^n$

Example 1. Suppose $A \in \mathbb{R}^n$,

$$p_T(x) = \left(\frac{1}{2\pi T}\right)^{n/2} e^{-\frac{|x|^2}{2T}} \; \forall \, x \in \mathbb{R}^n, \text{ and}$$

$$W_A(x) = -(\partial_A \ln p_T)(x) = \frac{x \cdot A}{T}.$$

Then

$$\int_{\mathbb{R}^n} e^{W_A(x)} p_T(x) \, dx = e^{\frac{|A|^2}{2T}},$$

and for all $y \in \mathbb{R}^n$,

$$\left(\int_{\mathbb{R}^n} \left[\frac{p_T(x-y)}{p_T(x)}\right]^p p_T(x) \, dx\right)^{1/p} = \exp\left(\frac{(p-1)}{2T} d^2(e,y)\right). \tag{1}$$

Proof: (Proof of Eq. (1).)

$$\begin{aligned}\int_{\mathbb{R}^n} \left[\frac{p_T(x-y)}{p_T(x)}\right]^p p_T(x) \, dx &= \int_{\mathbb{R}^n} \exp\left(\frac{p}{T} x \cdot y - \frac{p}{2T}|y|^2\right) p_T(x) \, dx \\ &= \exp\left(\frac{p^2 - p}{2T}|y|^2\right).\end{aligned}$$

Q.E.D.

Application (Cameron-Martin Theorem)

Theorem 1. *Let* (W, H, μ) *be an abstract Wiener space and* $h \in H$. *Then there exists* $Z_h \in \cap_{p'<\infty} L^{p'}(\mu)$ *such that, for all measurable,* $f : W \to [0, \infty)$,

$$\int_W f(h + w)\, d\mu(w) = \int_W f(w)\, Z_h(w)\, d\mu(w)$$

and

$$\|Z_h\|_{p'} \le \exp\left(\frac{p' - 1}{2} \|h\|_H^2\right).$$

Proof:

- Let $f : W \to [0, \infty)$ be a bounded and continuous function.
- Let $\{e_j\}_{j=1}^{\infty} \subset H_* \cong W^*$ be an orthonormal basis for H, and

$$P_n(w) = \sum_{j=1}^{n} \langle w, e_j \rangle_H e_j \text{ with } e_j \in H_*.$$

Then;

$$
\begin{aligned}
\int_W f \circ P_n (h+w)\, d\mu(w) &= \int_{P_n H} f(P_n h + x)\, p_1^{P_n H}(x)\, dx \\
&= \int_{P_n H} f(x)\, p_1^{P_n H}(x - P_n h)\, dx \\
&= \int_{P_n H} f(x)\, \frac{p_1^{P_n H}(x - P_n h)}{p_1^{P_n H}(x)} p_1^{P_n H}(x)\, dx \\
&\le \|f\|_{L^p\left(p_1^{P_n H}(x)dx\right)} \cdot \left\| \frac{p_1^{P_n H}(\cdot - P_n h)}{p_1^{P_n H}(\cdot)} \right\|_{L^{p'}\left(p_1^{P_n H}(x)dx\right)} \\
&\le \|f\|_{L^p\left(p_1^{P_n H}(x)dx\right)} \exp\left(\frac{p'-1}{2} \|P_n h\|_H^2\right).
\end{aligned}
$$

Now let $n \to \infty$ to find

$$
\int_W f(h+w)\, d\mu(w) \le \|f\|_{L^p(\mu)} \exp\left(\frac{p'-1}{2} \|h\|_H^2\right).
$$

Since, $L^{p*} \cong L^{p'}$, there exists $Z_h \in L^{p'}(\mu)$ such that

$$\int_W f(h+w)\, d\mu(w) = \int_W f(w)\, Z_h(w)\, d\mu(w)$$

and

$$\|Z_h\|_{p'} \le \exp\left(\frac{p'-1}{2}\|h\|_H^2\right).$$

Q.E.D.

Setup for Extending to Lie Groups

- $G =$ a connected finite dimensional uni-modular Lie group.
- $\mathfrak{g} = \mathrm{Lie}\,(G)$
- $(\cdot,\cdot) = (\cdot,\cdot)_{\mathfrak{g}}$ – an inner product on $\mathfrak{g}$, $|A|_{\mathfrak{g}} := \sqrt{(A,A)}$ for all $A \in \mathfrak{g}$.
- Use $(\cdot,\cdot)_{\mathfrak{g}}$ to make a left invariant Riemannian metric on G.
- $d\,(x,y)$ is the associated Riemannian distance function
- For $A \in \mathfrak{g}$ let $\tilde{A}$ denote the unique left invariant vector field on G such that $\tilde{A}\,(e) = A \in \mathfrak{g}$.
- $\Delta = \sum_{i=1}^{\dim \mathfrak{g}} \tilde{A}_i^2$ where $\{A_i\}_{i=1}^{\dim \mathfrak{g}}$ is an orthonormal basis for $\mathfrak{g}$.

Example

Let $G = \mathbb{R}^3$ with the multiplication law,

$$(x, y, z)(x', y', z') := \left(x + x', y + y', z + z' + \frac{1}{2}(xy' - yx') \right).$$

- A basis of left invariant vector fields is;

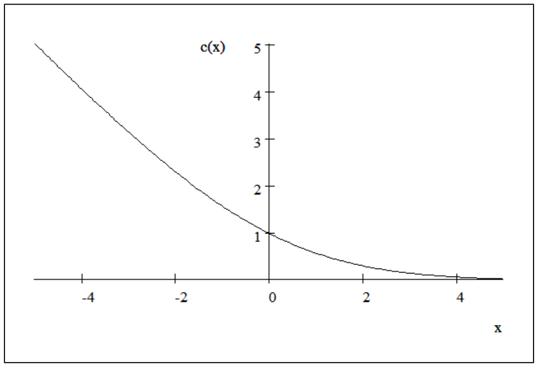
$$\begin{aligned} \tilde{A}_1 &= \frac{\partial}{\partial x} - \frac{1}{2} y \frac{\partial}{\partial z}, \\ \tilde{A}_2 &= \frac{\partial}{\partial y} + \frac{1}{2} x \frac{\partial}{\partial z} \\ \tilde{A}_3 &= [\tilde{A}_1, \tilde{A}_2] = \frac{\partial}{\partial z} \end{aligned}$$

- The Laplacian is then

$$\begin{aligned} \Delta &= \tilde{A}_1^2 + \tilde{A}_2^2 + \tilde{A}_3^2 \\ &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \frac{\partial}{\partial z} + \left(\frac{x^2 + y^2}{4} + 1 \right) \frac{\partial^2}{\partial z^2} \end{aligned}$$

Definition 2. Let $c(0) = 1$ and for $x \neq 0$,

$$c(x) = \frac{x}{e^x - 1}. \tag{2}$$



Theorem 3.1 (*Heat Kernel Jacobian Estimate.* Also see [Driver, 1997b]). Let $T > 0$,

$$p_T(x) = \left(e^{T\Delta/2}\delta_e\right)(x),$$

and $\mathrm{Ric} \geq kId$. Then for every $y \in G$ and $p \in [1, \infty)$,

$$\boxed{\left(\int_G \left[\frac{p_T(xy^{-1})}{p_T(x)}\right]^p p_T(x)\,dx\right)^{1/p} \leq \exp\left(\frac{c(kT)(p-1)}{2T} d^2(e, y)\right).} \tag{3}$$

Definition 3.2. For $A \in \mathfrak{g}$, let

$$W_A(x) := -\left(\tilde{A} \ln p_T\right)(x)$$

or equivalently,

$$\int_G \tilde{A} f(x)\, p_T(x)\, dx = \int_G f(x)\, W_A(x)\, p_T(x)\, dx \text{ for all } f \in C_c^\infty(G).$$

The proof of Theorem 3.1 relies on the following key exponential integrability estimate for W_A.

Theorem 3.3 (*Exponential Integrability*)**.** Continuing the notation in Theorem 3.1 and in particular let $c_T(k)$ be as in Eq. (2). Then for every $A \in \mathfrak{g}$,

$$\boxed{\int_G e^{W_A(x)} p_T(x)\, dx \le \exp\left(\frac{c(kT)}{2T} |A|_{\mathfrak{g}}^2\right).} \tag{4}$$

Applications

Let $d\nu_T(x) = p_T(x)\, dx$ be the heat kernel measure.

1. *Harmonic Pointwise Upper Bounds:* If $f \in L^p(\nu_T)$ and f is harmonic, then $P_T f = f$ and
$$|f(y)| \leq \|f\|_{L^p(\nu)} \exp\left(\frac{c(kT)}{2T(p-1)} d^2(e,y)\right). \tag{5}$$

2. *Holomorphic Pointwise Upper Bounds:* Eq. (5) holds if f is holomorphic but the constant is certainly not optimal. If $p = 2$, one can replace $\frac{c(kT)}{2T}$ by $\frac{1}{4T}$.

3. *Quasi-Invariance in Infinite Dimensions:*

   (a) Certain Projective Groups like Loop and Path groups, see for example, [Driver, 1997c] and [Fang, 1999]. Also see [Airault & Malliavin, 2006, JFA 241].

   (b) Certain infinite dimensional Nilpotent groups like an infinite dimensional Heisenberg groups, see Masha Gordina's talk.

Derivative Formulas

Theorem 5.1 (*Divergence Representation Formula I*). Assume the Ricci curvature, Ric, on M is bounded. Let $T > 0$ and $\tilde{\ell}$ be a continuous real-valued process $\tilde{\ell}_0 = 0$, $\tilde{\ell}_T = 1$, and there exists a non-random constant, $C < \infty$, such that

$$\int_0^T \left| \frac{d}{d\tau} \tilde{\ell}_\tau \right| d\tau \leq C. \tag{6}$$

Then for every C^1 – vector field, Y, on M with compact support the following identity holds

$$\boxed{\mathbb{E}\left[\nabla \cdot Y\left(\Sigma_T^x(x)\right)\right] = \mathbb{E}\left[\left\langle Y(\Sigma_T^x(x)), //_T Q_T \int_0^T \tilde{\ell}_t' Q_t^{-1} db_t \right\rangle\right],} \tag{7}$$

where $\nabla \cdot Y$ denote the divergence of Y. Here, $\Sigma_t^x \in M$ is Brownian motion, $//_t$ is parallel translation, and Q solves

$$\dot{Q}_t = -\frac{1}{2} \mathrm{Ric}^{//_t} Q_t \quad \text{with } Q_0 = id_{T_x M}. \tag{8}$$

Proof: See [Driver & Thalmaier, 2001]. Q.E.D.

Similar manipulations to those in [Driver, 1997b], shows Theorem 5.1 may be rephrased, in the Lie group setting $(M = G)$ as;

Theorem 5.2 (*Divergence Representation Formula II*)**.** Let $A \in \mathfrak{g}$, $\ell \in C^1([0,T], \mathbb{R})$ with $\ell(0) = 0$ and $\ell(T) = 1$ and define;

$$\beta_t := \int_0^t L_{\Sigma_\tau^{-1}*} \delta \Sigma_\tau.$$

Then

$$\boxed{-\left(\tilde{A} \ln p_T\right)(x) = W_A(x) = \mathbb{E}\left[\left(A, \int_0^T \dot{\ell}(\tau) V_\tau d\overleftarrow{\beta}_\tau\right) \Big| \Sigma_T = x\right],} \tag{9}$$

where $\int_0^T \dot{\ell}(\tau) V_\tau d\overleftarrow{\beta}_\tau$ is a backwards Itô integral and V_t satisfies

$$dV_t = \frac{1}{2} V_t \operatorname{Ric}_e dt + V_t D_{\circ d\beta_t} \text{ with } V_T = Id.$$

Here,

$$D_A : \mathfrak{g} \longrightarrow \mathfrak{g} \text{ for all } A \in \mathfrak{g}$$

is the Levi-Civita connection as seen on $\mathfrak{g}$ and $\Sigma_t \in G$ is Bronwian motion starting at $e \in G$.

Lemma 5.3. Suppose $k \in \mathbb{R}$ is chosen so that $\operatorname{Ric} \geq kI$, then

$$|V_t^* A|^2 \leq |A|^2 e^{-k(T-t)} \text{ for all } A \in \mathfrak{g}. \tag{10}$$

Proof of Theorem 3.3

Theorem 6.1 (Theorem 3.3)**.**

$$\int_G e^{W_A(x)} p_T(x)\, dx \le \exp\left(\frac{c(kT)}{2T} |A|_{\mathfrak{g}}^2\right) \text{ for all } A \in \mathfrak{g}.$$

Proof: Let $\ell \in C^1([0,T],\mathbb{R})$ such that $\ell(0) = 0$ and $\ell(T) = 1$. From Theorem 5.2, Lemma 5.3, Jensen's inequality for conditional expectations, and a standard martingale argument (see the proof of Lemma 7.6 and especially Eq. 7.17 in [Driver, 1997a]), starting with

$$\boxed{W_A(x) = \mathbb{E}\left[\left(A, \int_0^T \dot{\ell}(\tau) V_\tau d\overleftarrow{\beta}_\tau\right)\middle| \Sigma_T = x\right],}$$

we have. . .

$$\begin{aligned}\int_G e^{W_A} d\nu_T &= \mathbb{E}\left[\exp\left(\mathbb{E}\left[\left(A, \int_0^T \dot{\ell}(\tau) V_\tau d\overleftarrow{\beta}_\tau\right) \middle| \sigma(\Sigma_T)\right]\right)\right] \\ \text{(Jensen)} &\leq \mathbb{E}\left[\mathbb{E}\left[\exp\left(\left(A, \int_0^T \dot{\ell}(\tau) V_\tau d\overleftarrow{\beta}_\tau\right)\right) \middle| \sigma(\Sigma_T)\right]\right] \\ &\leq \mathbb{E}\left[\exp\left(\left(A, \int_0^T \dot{\ell}(\tau) V_\tau d\overleftarrow{\beta}_\tau\right)\right)\right] \\ \text{(Mrtg. Est.)} &\leq \exp\left(\frac{1}{2}\left\|\int_0^T \dot{\ell}^2(\tau) \left|V_\tau^* A\right|^2 d\tau\right\|_{L^\infty(P)}\right) \\ \text{(Lemma 5.3)} &\leq \exp\left(\frac{|A|^2}{2}\int_0^T \dot{\ell}^2(\tau) e^{-k(T-\tau)} d\tau\right).\end{aligned}$$

The last expression is minimized by taking,

$$\ell(\tau) = \frac{e^{-k\tau} - 1}{e^{-kT} - 1}$$

and this gives;

$$\int_G e^{W_A} d\nu_T \leq \exp\left(\frac{1}{2T} c(kT) |A|^2\right).$$

Q.E.D.

Exponential Integrability –> Jacobian Estimate

Our next goal is to show that

Theorem 7.1 (Theorem 3.3)**.**

$$\int_G e^{W_A(x)} p_T(x)\,dx \le \exp\left(\frac{c(kT)}{2T}|A|_{\mathfrak{g}}^2\right) \text{ for all } A \in \mathfrak{g}.$$

$$\Downarrow$$

Theorem 7.2 (*Heat Kernel Jacobian Estimate.*)**.** Let $T > 0$,

$$p_T(x) = \left(e^{T\Delta/2}\delta_e\right)(x),$$

and $\mathrm{Ric} \ge kId$. Then for every $y \in G$ and $p \in [1, \infty)$,

$$\boxed{\left(\int_G \left[\frac{p_T(xy^{-1})}{p_T(x)}\right]^p p_T(x)\,dx\right)^{1/p} \le \exp\left(\frac{c(kT)(p-1)}{2T} d^2(e, y)\right).}$$

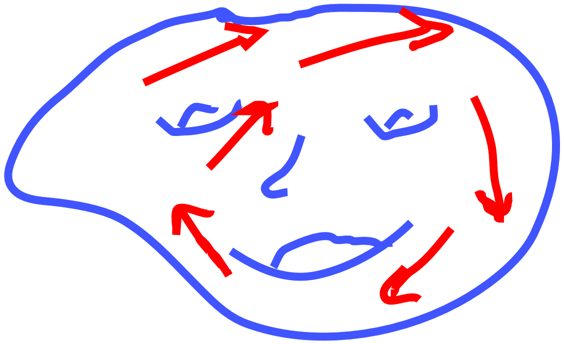
Some key ideas;

- We may find a path, $A_t \in \mathfrak{g}$ such that $S_1(x) = R_y(x) = xy$ where S_t solves the flow equation,
$$\frac{d}{dt}S_t(x) = \left(\tilde{A}_t S_t\right)(x) \text{ with } S_0(x) = x \,\forall\, x \in G.$$

- $d(x,y) = \inf\left\{\int_0^1 |A_t|_{\mathfrak{g}}\, dt : \text{ over all such } t \to A_t\right\}.$

- If we let $d\mu(x) := p_T(x)\,dx$, then for $y \in G$,
$$J_1(x) := \frac{d\mu \circ S_1^{-1}}{d\mu}(x) = \frac{p_T\left(xy^{-1}\right)}{p_T(x)}.$$

Jacobian Estimates Generalities



- Let M be a finite dimensional manifold,
- μ be a smooth positive probability measure on M
- $\|f\|_r := \left(\int_M |f|^r \, d\mu\right)^{1/r}$
- X_t be a time dependent vector field on M
- S_t denote the flow, i.e. $S_t(m)$ solves,

$$\frac{d}{dt}S_t(m) = X_t \circ S_t(m) \text{ with } S_0(m) = m \text{ for all } m \in M. \tag{11}$$

- Assume that X_t is forward complete.
- Let W_t be the μ – **divergence** of X_t, i.e.

$$\int_M X_t f \, d\mu = \int_M W_t f \, d\mu \text{ for all } f \in C_c^\infty(M).$$

- Define

$$\mu_t = (S_t)_* \mu = \mu \circ S_t^{-1} \text{ and } J_t = d\mu_t / d\mu.$$

The Galaz-Fontes, Gross, and Sontz Method

Notation 9.1. For any function, $W : M \to \mathbb{R}$,

$$\mathcal{B}(W) := \ln\left(\mu\left(e^W\right)\right) = \ln\left(\int_M e^W d\mu\right).$$

- The following is an extension of a theorem of Galaz-Fontes, Gross, and Sontz which extends the pioneering work of Cruzerio in this direction.

Theorem 9.2 (*Jacobian Estimate* [Galaz-Fontes *et al.* , 2001])**.** Let $p > 1$ and $r \in C([0,\tau],[1,\infty)) \cap C^1((0,\tau),(1,\infty))$ such that $r(0) = 1$, $r(\tau) = p$ and $\dot{r}(t) > 0$ for $0 < t < \tau$, then

$$\|J_\tau\|_{p'} \le e^{\Lambda(r)}, \tag{12}$$

where $p' := p/(p-1)$ is the conjugate exponent to p and

$$\Lambda(r) = \Lambda_X(r) := \int_0^\tau \frac{\dot{r}(t)}{r^2(t)} \mathcal{B}\left(\frac{r(t)}{\dot{r}(t)} W_t\right) dt. \tag{13}$$

- We will now use this theorem with

$$M = G \text{ and } d\mu(x) = p_T(x)\, dx.$$

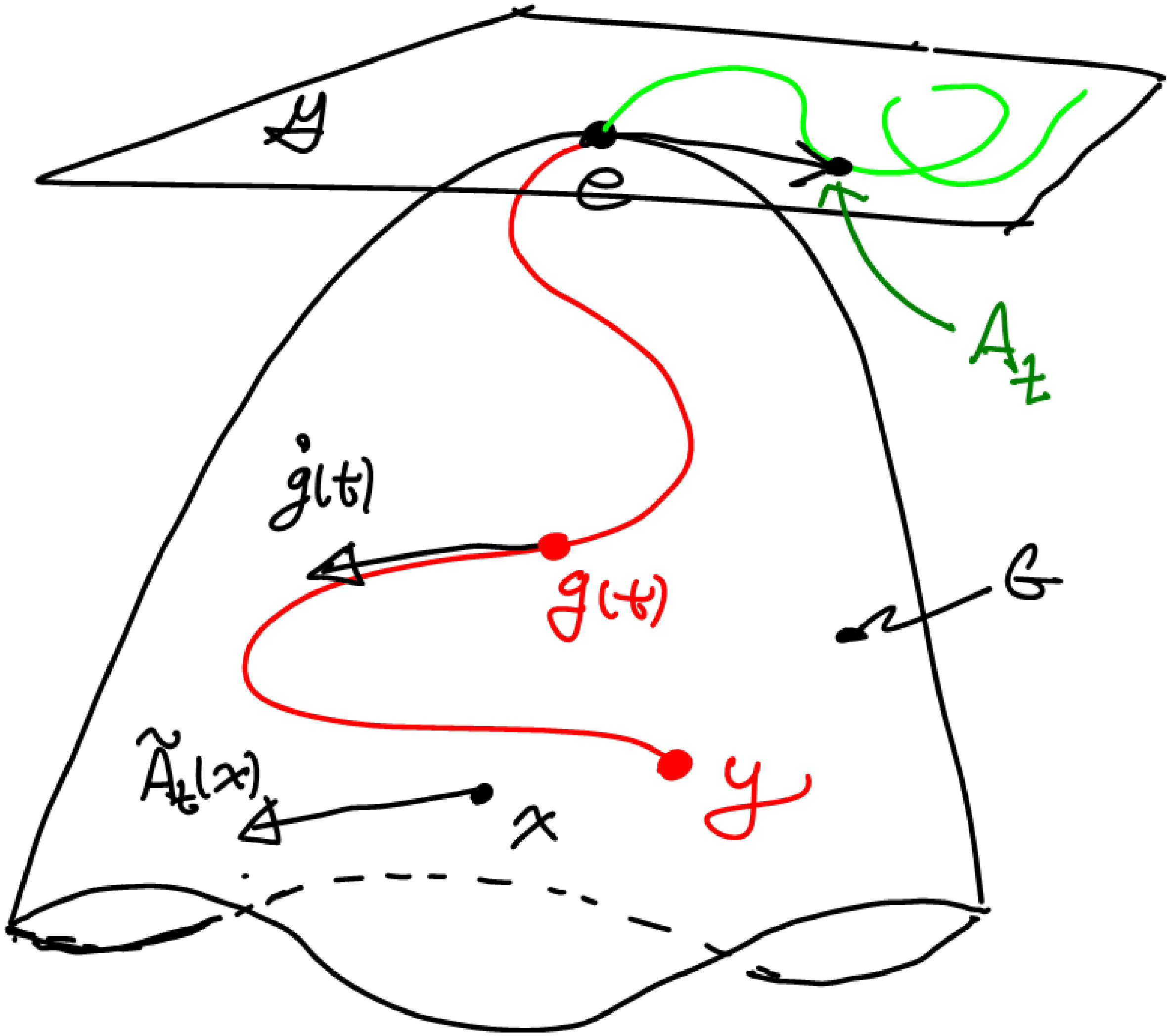
Proof of Theorem 3.1

Theorem 10.1 (Theorem 3.1)**.** If $\mathrm{Ric} \geq kI$, $y \in G$, and $p \in [1, \infty)$, then

$$\left(\int_G \left[\frac{p_T\left(xy^{-1}\right)}{p_T(x)} \right]^p p_T(x)\, dx \right)^{1/p} \leq \exp\left(\frac{c(kT)(p-1)}{2T} d^2(e, y) \right).$$

Proof: Let

- $g \in C^1([0,1], G)$ such that $g(0) = e \in G$ and $g(1) = y \in G$
- $A_t := L_{g(t)*}^{-1} \dot{g}(t) \in \mathfrak{g}$ and $X_t := \tilde{A}_t \in \Gamma(TG)$.
- So $S_t(x) = xg(t)$.
- $J_1(x) := \frac{d(S_1)_* \mu}{d\mu}(x) = \frac{p_T\left(xy^{-1}\right)}{p_T(x)}$
- $W_t = -\tilde{A}_t \ln p_T$ is the μ – divergence of $X_t = \tilde{A}_t$.
- $r(0) = 1$, $r(1) = p$ and $\dot{r}(t) > 0$ for $0 < t < 1$.



$$A_t = L_{g(t)*}^{-1}\dot{g}\,(t) \text{ and } \ell\,(g) = \int_0^1 |A_t|_{\mathfrak{g}}\, dt.$$

1. By exponential integrability Theorem 3.3,

$$\mathcal{B}(\lambda W_t) = \ln\left(\int_G e^{\lambda W_t} d\mu\right) \le c\lambda^2 |A_t|_{\mathfrak{g}}^2 .$$

2. By extended Galaz-Fontes, Gross, Sontz Theorem 9.2,

$$\left(\int_G \left[\frac{p_T(xy^{-1})}{p_T(x)}\right]^{p'} p_T(x)\,dx\right)^{1/p'} = \|J_1\|_{p'} \le e^{\Lambda(r)}, \tag{14}$$

   where

$$\Lambda(r) = \int_0^1 \frac{\dot{r}(t)}{r^2(t)} \mathcal{B}\left(\frac{r(t)}{\dot{r}(t)} W_t\right) dt$$
$$\le c \int_0^1 \frac{\dot{r}(t)}{r^2(t)} \frac{r^2(t)}{\dot{r}^2(t)} |A_t|_{\mathfrak{g}}^2\, dt = c\int_0^1 \frac{|A_t|_{\mathfrak{g}}^2}{\dot{r}(t)} dt,$$

3. $\Lambda(r)$ is minimized by taking

$$r(t) = 1 + \kappa \int_0^t |A_\tau|_{\mathfrak{g}}\, d\tau$$

   with

$$\kappa = (p-1)\left(\int_0^1 |A_\tau|_{\mathfrak{g}}\, d\tau\right)^{-1}.$$

With this choice of r,

$$\Lambda(r) = \frac{c(kT)(p'-1)}{2T}\left(\int_0^1 |A_t|_{\mathfrak{g}}\, dt\right)^2 \tag{15}$$

$$= \frac{c(kT)(p'-1)}{2T}\ell^2(g) \tag{16}$$

4. Combining Eqs. (14) and Eq. (15) and then optimizing over the paths, g, joining e to y gives the result (with $p \to p'$).

Q.E.D.

A Conjecture

Conjecture 11.1. Let (M, g) be a complete Riemannian manifold such that $\mathrm{Ric} \geq kI$ for some $k \in \mathbb{R}$. Then for every $y, z \in M$ and $p \in [1, \infty)$,

$$\left(\int_G \left[\frac{p_T(y, x)}{p_T(z, x)}\right]^p p_T(z, x)\, dx\right)^{1/p} \leq \exp\left(\frac{c(kT)(p-1)}{2T} d^2(y, z)\right) \tag{17}$$

where $c = c_T(k)$ is defined as in Eq. (2), $p_t(x, y)$ is the heat kernel on M and $d(y, z)$ is the Riemannian distance from x to y for $x, y \in M$.

On the proof of the Jacobian Estimates

Lemma 12.1 (*Young's Inequality*)**.** Let $W \in L^1(\mu)$ and $f \geq 0$ be a bounded measurable function. Then, for all $s > 0$,

$$\int_M W f d\mu \leq s \int_M f \ln \frac{f}{\mu(f)} d\mu + s\mathcal{B}(W/s) \int_M f d\mu \tag{18}$$

where

$$\mathcal{B}(W) := \ln\left(\mu\left(e^W\right)\right) = \ln\left(\int_M e^W d\mu\right).$$

Proof: Applying Young's inequality ($xy \leq e^x + y \ln y - y$ for $x \in \mathbb{R}$ and $y \geq 0$) with $x = W$ and $y = f$ and then integrating gives;

$$\int_M W f d\mu \leq \int_M e^W d\mu + \int_M [f \ln f - f] \, d\mu.$$

Replacing f by λf with $\lambda > 0$ and the optimizing the result in λ implies

$$\int_M W f d\mu \leq \int_M f \ln \frac{f}{\mu(f)} d\mu + \mathcal{B}(W) \int_M f d\mu. \tag{19}$$

Finish the proof by replacing W by W/s. Q.E.D.

Definition 12.2. The μ*–divergence* of a smooth vector field, X, on M is the function $W = W_X$ defined by

$$\int_M X\varphi d\mu = \int_M \varphi W d\mu, \text{ for all } \varphi \in C_c^1(M).$$

Proposition 12.3. Let $W_t := W_{X_t}$ be the μ*–divergence* of X_t, $h \in C^1\left(M, [0,\infty)\right)$, $h_t := h \circ S_t^{-1}$, and $r \in C^1\left((0,\tau), (1,\infty)\right)$. Then for any $s > 0$ we have

$$\frac{d}{dt} \ln \|h_t\|_{r(t)} \geq \left(\frac{\dot{r}}{r} - s\right) \int_M \frac{h_t^r}{\|h_t\|_r^r} \left(\ln \frac{h_t}{\|h_t\|_r}\right) d\mu - \frac{s}{r} \mathcal{B}\left(s^{-1} W_t\right). \tag{20}$$

Proof: Let $r = r(t)$ and $\dot{r} = \dot{r}(t)$, then by pure calculation;

$$\frac{d}{dt} \ln \|h_t\|_{r(t)} = \frac{\dot{r}}{r} \int_M \frac{h_t^r}{\|h_t\|_r^r} \left(\ln \frac{h_t}{\|h_t\|_r}\right) d\mu - \frac{1}{r} \int_M W_t \Bigg(\overbrace{\frac{h_t^r}{\|h_t\|_r^r}}^{f}\Bigg) d\mu.$$

Using the estimate in Eq. (18) with $W = W_t$ and $f = \frac{h_t^r}{\|h_t\|_r^r}$ then gives Eq. (20). Q.E.D.

Remark 3. Taking $s = \dot{r}/r$ in Eq. (20) shows;

$$\frac{d}{dt} \ln \|h_t\|_{r(t)} \geq -\frac{\dot{r}}{r^2} \mathcal{B}\left(s^{-1} W_t\right). \tag{21}$$

The following theorem is the extension of [Galaz-Fontes *et al.* , 2001, Theorem 2.14] from time independent vector fields to time dependent vector fields.

Theorem 12.4 (*Jacobian Estimate*)**.** Let $p > 1$ and $r \in C([0,\tau],[1,\infty)) \cap C^1((0,\tau),(1,\infty))$ such that $r(0) = 1$, $r(\tau) = p$ and $\dot{r}(t) > 0$ for $0 < t < \tau$, then

$$\|J_\tau\|_{p'} \le e^{\Lambda(r)}, \tag{22}$$

where $p' := p/(p-1)$ is the conjugate exponent to p and

$$\Lambda(r) = \Lambda_X(r) := \int_0^\tau \frac{\dot{r}(t)}{r^2(t)} \mathcal{B}\left(\frac{r(t)}{\dot{r}(t)} W_t\right) dt. \tag{23}$$

Proof: Integrating Eq. (21);

$$\frac{d}{dt} \ln \|h_t\|_{r(t)} \ge -\frac{\dot{r}}{r^2} \mathcal{B}\left(s^{-1} W_t\right),$$

implies

$$\left\|h \circ S_\tau^{-1}\right\|_p = \|h_\tau\|_p \ge \|h\|_1 e^{-\Lambda(r)}.$$

Letting $h \to h \circ S_\tau$ then implies,

$$\int_M h J_\tau d\mu = \|h \circ S_\tau\|_1 \le \|h\|_p e^{\Lambda(r)}. \tag{24}$$

By the converse to Hölder's inequality, it now follows that Eq. (22) is valid. Q.E.D.

References

[Airault & Malliavin, 2006] Airault, Hélène, & Malliavin, Paul. 2006. Quasi-invariance of Brownian measures on the group of circle homeomorphisms and infinite-dimensional Riemannian geometry. *J. funct. anal.*, **241**(1), 99–142.

[Driver, 1997a] Driver, Bruce K. 1997a. Integration by parts and quasi-invariance for heat kernel measures on loop groups. *J. funct. anal.*, **149**(2), 470–547.

[Driver, 1997b] Driver, Bruce K. 1997b. Integration by parts for heat kernel measures revisited. *J. math. pures appl. (9)*, **76**(8), 703–737.

[Driver & Thalmaier, 2001] Driver, Bruce K., & Thalmaier, Anton. 2001. Heat equation derivative formulas for vector bundles. *J. funct. anal.*, **183**(1), 42–108.

[Fang, 1999] Fang, Shizan. 1999. Integration by parts for heat measures over loop groups. *J. math. pures appl. (9)*, **78**(9), 877–894.

[Galaz-Fontes *et al.* , 2001] Galaz-Fontes, Fernando, Gross, Leonard, & Sontz, Stephen Bruce. 2001. Reverse hypercontractivity over manifolds. *Ark. mat.*, **39**(2), 283–309.

A Calculus of Variation Lemma

Lemma 14.1. Let $k \in \mathbb{R}$ and $T > 0$, then

$$\inf \left\{ \int_0^T \dot{\ell}^2 (\tau) e^{-k(T-\tau)} d\tau \right\} \leq \frac{k}{e^{kT} - 1} = \frac{c(kT)}{T} \tag{25}$$

where the infimum is taken over all $\ell \in C^1([0, T], \mathbb{R})$ such that $\ell(0) = 0$ and $\ell(T) = 1$.

Proof: By a simple calculus of variation argument, $\ell \in C^1([0, T], \mathbb{R})$ with $\ell(0) = 0$ and $\ell(T) = 1$ is a critical point for the function,

$$K(\ell) := \int_0^T \dot{\ell}^2 (\tau) e^{-k(T-\tau)} d\tau, \tag{26}$$

iff $\dot{\ell}(\tau) e^{k\tau}$ is constant in τ. This constraint and the boundary conditions imply that K has a unique critical point at

$$\ell_c(\tau) = \frac{e^{-k\tau} - 1}{e^{-kT} - 1}.$$

Plugging this value of ℓ_c into K then shows $K(\ell_c) = k\left(1 - e^{-kT}\right)^{-1}$ from which Eq. (25) follows. Q.E.D.