Gradient estimates and Harnack inequality on non compact manifold

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Complex stochastic systems: Discrete vs continuous
W2 Stochastic calculus on manifolds, graphs and random structures

joint works with

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An elementary lemma

Lemma 1

$(E, \mathcal{F}, \tilde{\mu})$ measured space such that $\tilde{\mu}(E) < \infty$.
$f \in L^1(\tilde{\mu})$ positive satisfying $\tilde{\mu}(f) > 0$
ψ mesurable satisfying $\psi f \in L^1(\tilde{\mu})$.
Then

$$\int_E \psi f \, d\tilde{\mu} \le \int_E f \log \frac{f}{\tilde{\mu}(f)} \, d\tilde{\mu} + \tilde{\mu}(f) \log \int_E e^{\psi} \, d\tilde{\mu}.$$

Horizontal diffusion in C^1 path space

M a complete Riemannian manifold
$L = \frac{1}{2}\Delta + Z$, $Z \in \Gamma(TM)$.
If Y is a L-diffusion, denote by $W(Y)_t : T_{Y_0}M \to T_{Y_t}M$ the deformed parallel translation along Y:

$$DW(Y) = \left(-\frac{1}{2}\mathrm{Ric}^\sharp(W(Y)) + \nabla_{W(Y)}Z\right) dt.$$

Theorem 1

$u \mapsto \varphi(u)$ a C^1 path in M, defined on $[0, \infty[$
X^0 a L-diffusion started at $\varphi(0)$.
There exists a unique family $u \mapsto (X_t(u))_{t \geq 0}$ of L-diffusions, a.s. continuous in (t, u) and C^1 in u, satisfying $X(0) = X^0$, $X_0(u) = \varphi(u)$, and

$$\partial_u X_t(u) = W(X(u))_t(\dot{\varphi}(u)),$$

Moreover

$$dX_t(u) = P^{X(\cdot)}_{t,0,u} d_m X^0_t + Z_{X_t(u)}\, dt$$

with $v \mapsto P^{X(\cdot)}_{t,0,v}$ parallel transport along $v \mapsto X_t(v)$.

Gradient estimates using coupling

- M non compact Riemannian manifold
- $L = \frac{1}{2}\Delta + Z, \qquad Z = \mathrm{grad}\, V,\ V \in C^2(M)$
- P_t Dirichlet semigroup associated with L :

$$P_t f(x) = \mathbb{E}\left[f(X_t(x))1_{\{t<\xi(x)\}}\right]$$

where $X(x)$ is a diffusion with generator L satisfying $X_0(x) = x$, explosion time $\xi(x)$.

- D a C^2 relatively compact domain in M
- P_t^D Dirichlet semigroup :

$$P_t^D f(x) = \mathbb{E}\left[f(X_t(x))1_{\{t<\tau(x)\}}\right]$$

where $\tau(x)$ is the exit time of D.

[Qi S. Zhang 06] : $f \in B_b^+$ (nonnegative measurable functions)

$$\left\| \mathrm{grad}\, P_t^D f(x) \right\| \leq C(x,t) P_t^D f(x)$$

where $C(x,t)$ is a locally bounded function on $D \times]0, \infty[$.

Consequence :
If $f \in B_b^+$, $t > 0$, $x, y \in D$,

$$P_t^D f(x) \leq \tilde{C}(x,y,t) P_t^D f(y)$$

where $\tilde{C}$ is locally bounded on $D \times D \times]0, \infty[$

Aim: obtain similar estimates on M

Remark: these estimates are wrong in general, take for instance $M = \mathbb{R}^d$ and $L = \frac{1}{2}\Delta$.

The main result

Theorem 2

There exists a continuous function $F :]0,1] \times M \to \mathbb{R}_+$ such that $\forall \delta > 0$, $x \in M$, $t > 0$, $f \in B_b^+$,

$$\|\mathrm{grad}\, P_t f(x)\| \leq \delta \left[P_t \left(f \log \frac{f}{P_t f(x)} \right) \right](x) + \left(F(\delta \wedge 1, x) \left(\frac{1}{\delta(t \wedge 1)} + 1 \right) + \frac{2\delta}{\mathrm{e}} \right) P_t f(x).$$

Case $\mathrm{Ric} - \mathrm{Hess}\, V$ bounded

Let $h_s = 1 - \frac{s}{t}$, $v \in T_xM$, $\varphi(u) = \exp(uv)$. then for $\ell > 0$

$$\begin{aligned} d\left(X_s(\ell h_s)\right) &= (dX_s)(\ell h_s) + \partial X_s(\ell h_s)\ell \dot{h}_s\, ds \\ &= (dX_s)(\ell h_s) + W\left(X(\ell h_s)\right)\dot{\varphi}(\ell h_s)\ell \dot{h}_s\, ds. \end{aligned}$$

The last term in the right is uniformly bounded. Let

$$N_s^\ell = -\int_0^s \left\langle W(X(\ell h_r v))\dot{\varphi}(\ell h_r)\ell \dot{h}_r, d_m X_r(\ell h_r)\right\rangle,$$

$$R_s^\ell = \mathcal{E}(N^\ell)_s, \qquad \mathbb{Q}^\ell = R^\ell \cdot \mathbb{P}.$$

Under $\mathbb{Q}^\ell$, $X_s(\ell h_s)$ is a L-diffusion started at $\exp(\ell v)$, satisfying $X_t(\ell h_t) = X_t^0$.

Consequence

$$P_t f(\exp(\ell v)) = \mathbb{E}\left[f(X_t^0) R_t^\ell\right].$$

Choosing $v = \dfrac{\mathrm{grad}\, P_t f(x)}{\|\mathrm{grad}\, P_t f(x)\|}$, differentiating w.r.t. to ℓ at $\ell = 0$, one obtains

$$\mathrm{grad}\, P_t f(x) = \mathbb{E}\left[f(X_t^0) \int_0^t -\left\langle W(X^0)(v\dot{h}),\ d_m X^0 \right\rangle\right].$$

Then applying the elementary lemma to $\delta f(X_t^0)$ and $-\dfrac{1}{\delta}\displaystyle\int_0^t \left\langle W(X^0)(v\dot{h}),\ d_m X^0 \right\rangle$ one obtains the wanted result

The proof for $P_t^D f$

Let $c \in]0,1[$ and $h_s = \left(1 - \frac{s}{ct}\right)_+$. Then

$$\frac{1}{\ell}\left(P_t^D f(\exp(\ell v)) - P_t^D f(x)\right) \leq \mathbb{E}\left[f(X_t^0) 1_{\{t<\tau(x)\}} \frac{1}{\ell}\left(R_t^\ell - 1\right)\right] + \frac{1}{\ell}\mathbb{E}\left[f(X_t^\ell) R_t^\ell 1_{\{\tau(x)\leq ct<\tau^\ell\}}\right].$$

Then we prove that for c small

$$\frac{1}{\ell}\mathbb{E}\left[f(X_t^\ell) R_t^\ell 1_{\{\tau(x)\leq ct<\tau^\ell\}}\right] \leq C P_t^D f(x).$$

The general case

Lemma 2

For $x \in D$, let $h_x(s,z)$ the density of $(\tau(x), X_{\tau(x)}(x))$. Then

$$P_t f(x) = P_t^D f(x) + \int_{]0,t]\times\partial D} P_{t-s} f(z) h_x(s,z)\, ds \nu(dz).$$

Proof: strong Markov property

consequence :

$$\mathrm{grad}\, P_t f(x) = \mathrm{grad}\, P_t^D f(x) + \int_{]0,t]\times\partial D} P_{t-s} f(z)\, \mathrm{grad} \log h_\cdot(s,z) h_x(s,z) ds \nu(dz).$$

then apply the elementary lemma to the second integral, with the measure on $]0,t] \times \partial D$

$$\tilde{\mu} = h_x(s,z) ds \nu(dz) \quad \text{with total mass} \quad \mathbb{P}(\tau(x) \le t < \xi(x))$$

and the functions $\delta P_{t-s} f(z)$, $\frac{1}{\delta} \| \mathrm{grad} \log h_\cdot(s,z) \|$. We obtain for the secon term

$$\delta \mathbb{E}\left[f \log \frac{f(X_t)}{l} 1_{\tau(x)\le t<\xi(x)} \right] + l \log \mathbb{E}\left[1_{\tau(x)\le t<\xi(x)} e^{\frac{1}{\delta}\| \mathrm{grad} \log h_x(\tau, X_\tau)\|} \right]$$

with $l = \mathbb{E}\left[f(X_t) 1_{\tau(x)\le t<\xi(x)} \right]$.
Some additional work is needed to estimate
$\log \mathbb{E}\left[1_{\tau(x)\le t<\xi(x)} e^{\frac{1}{\delta}\| \mathrm{grad} \log h_x(\tau, X_\tau)\|} \right]$.

Harnack inequality and heat kernel estimate

Harnack inequality

Corollary 1

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There exists a continuous $C :]1,\infty[\times M \times M \to \mathbb{R}_+$, such that $\forall \alpha > 1$, $t > 0$, $x, y \in M$, $f \in B_b^+$

$$(P_t f(x))^\alpha \le P_t f^\alpha(y) \exp\left[\frac{2(\alpha-1)}{e} + \alpha C(\alpha,x,y)\left(\frac{\alpha\rho^2(x,y)}{(\alpha-1)(t\wedge 1)} + \rho(x,y)\right)\right]$$

where $\rho(x, y)$ is the Riemannian distance from x to y.

Proof

Let γ_s be the minimal geodesic from x to y in time 1, Theorem 2 yields a bound below for

$$\frac{d}{ds}\log\left(P_t f^{1+s(\alpha-1)}\right)^{\frac{\alpha}{1+s(\alpha-1)}}(\gamma_s).$$

Integrating from 0 to 1 gives the result.

Heat kernel estimate

Corollary 2

,
$\forall \delta > 2, \exists C_\delta : [0, \infty[\times M \to \mathbb{R}_+, \forall x, y \in M, \ t \in]0, 1[,$

$$p_t(x, y) \leq \frac{\exp\left[\frac{-\rho(x,y)^2}{\delta t} + C_\delta(t, x) + C_\delta(t, y)\right]}{\sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}},$$

with $\mu(dx) = e^{V(x)}\, dx$.