A disorder model and its use to determine exit and entry strategies in various financial bubble markets

M. V. Zhitlukhin   W. T. Ziemba

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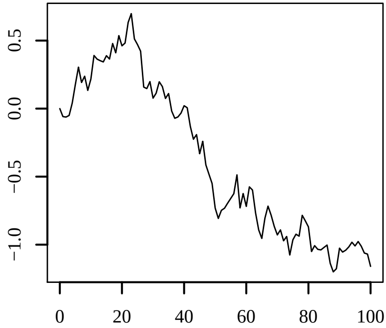
A disorder of a random sequence is an unknown moment of time when its probabilistic characteristics changes.

For example, the mean or the variance of a random sequence changes.



In general, our aim is to detect a disorder in a sequence of asset prices while sequentially observing it.

Gaussian random walk with a disorder

Let $S_0, S_1, S_2, \dots$ be a random sequence representing the prices of an asset at the moments of time $t = 0, 1, 2, \dots$.

Assume that the log-returns are distributed as follows:

$$\log \frac{S_t}{S_{t-1}} = \begin{cases} \mu_1 + \sigma_1 \xi_t, & t < \theta \\ \mu_2 + \sigma_2 \xi_t, & t \geqslant \theta \end{cases}$$

where

$\mu_1, \mu_2, \sigma_1, \sigma_2$ are known parameters,
ξ_i are i.i.d. $\mathcal{N}(0,1)$ random variables,
$\theta \in \{1, 2, \dots, T+1\}$ is the moment of disorder of the price sequence.

We assume that θ is an unobservable random variable independent of ξ_t with a known distribution $\mathrm{P}(\theta = t) = p_t$.

(*Remark:* p_1 is the probability that $\mu = \mu_2$, $\sigma = \sigma_2$ from the beginning; p_{T+1} is the probability that $\mu = \mu_1$, $\sigma = \sigma_1$ until the time $t = T$.)

The question we consider:

when is it optimal to sell the asset in the above model?

By definition, the moment τ when one sells the asset should be a stopping time of the sequence S, i. e.

$$\{\tau \leqslant t\} \in \sigma(S_u; u \leqslant t) \quad \text{for any } t = 1, 2, \ldots,$$

which means that a decision to sell the asset should be based only on the price history up to the present moment of time.

We consider two approaches for choosing the optimal stopping time:

1. Maximizing the expected utility from selling the asset at time τ; (computationally difficult)
2. Stopping as close as possible to θ (computationally easy).

Maximizing the expected utility

Let $\{U_\alpha(x)\}_\alpha$, $a \in (-\infty, 1]$ be the family of utility functions:

$$U_\alpha(x) = x^\alpha, \alpha \in (0,1], \quad U_0(x) = \log(x), \quad U_\alpha(x) = -x^\alpha, \alpha < 0.$$

We consider the following optimal stopping problems for $\alpha \leqslant 1$:

$$V^\alpha = \sup_{\tau \leqslant T} \mathrm{E} U_\alpha(S_\tau).$$

The problems consist in finding the stopping times τ_α^* at which the suprema are attained.

Based on a recent result by A. N. Shiryaev & M. V. Zhitlukhin; submitted to *Theory of Probability and Its Applications*.

Define

$$X_t = \log \frac{S_t}{S_{t-1}}, \qquad t = 1, \dots, T.$$

Introduce the Shiryaev–Roberts statistic $\psi = (\psi_t)_{t \geqslant 0}$:

$$\psi_0 = 0, \quad \psi_t = (p_t + \psi_{t-1}) \cdot \tfrac{\sigma^1}{\sigma^2} \exp\Big(\tfrac{(X_t - \mu_1)^2}{2\sigma_1^2} - \tfrac{(X_t - \mu_2)^2}{2\sigma_2^2} \Big).$$

Theorem. *The optimal stopping time in problem V^α is given by*

$$\tau_\alpha^* = \inf\{0 \leqslant t \leqslant T : \psi_t \geqslant a_\alpha(t)\},$$

where

$$a_\alpha(t) = \inf\{x \geqslant 0 : V_t^\alpha(x) = 0\}$$

for the family of functions $V_0^\alpha, V_1^\alpha, \dots, V_T^\alpha$, which are non-decreasing and can be found recurrently as follows:

$\dots\dots$

For $\alpha = 0$*:*

$$V_T^0(x) = 0 \text{ for all } x \geqslant 0;$$

$$V_t^0(x) = \max\big\{0,\ \mu_1(1 - G(t+1)) + \mu_2(x + p_{t+1}) + f^0(t,x)\big\},$$

$$\text{where} \quad f^0(t,x) = \int_{\mathbb{R}} V_{t+1}^0\Big[\Big(p_{t+1} + x\Big) \cdot \frac{\sigma_1}{\sigma_2} \exp\Big(\tfrac{(z-\mu_1)^2}{2\sigma_1^2} - \tfrac{(z-\mu_2)^2}{2\sigma_2^2}\Big)\Big] \times \frac{1}{\sigma_1\sqrt{2\pi}} \exp\Big(-\tfrac{(z-\mu_1)^2}{2\sigma_1^2}\Big) dz$$

For $\alpha \neq 0$*:*

$$V_T^\alpha(x) = 0 \text{ for all } x \geqslant 0;$$

$$V_t^\alpha(x) = \max\big\{0,\ \operatorname{sgn}(\alpha) \cdot \beta^t\big[(\beta - 1)(1 - G(t+1)) + (\gamma - 1)(p_{t+1} + x)\big] + f^\alpha(t,x)\big\},$$

$$\text{where} \quad f^\alpha(t,x) = \int_{\mathbb{R}} V_{t+1}^\alpha\Big[\Big(p_{t+1} + x\Big) \cdot \frac{\sigma_1}{\sigma_2} \exp\Big(\tfrac{(z-\mu_1)^2}{2\sigma_1^2} - \tfrac{(z-\mu_2)^2}{2\sigma_2^2}\Big)\Big] \times \frac{1}{\sigma_1\sqrt{2\pi}} \exp\Big(-\tfrac{(z-\mu_1-\alpha\sigma_1^2)^2}{2\sigma_1^2}\Big) dz$$

$\dots\dots$

with the constants

$$\beta = \exp\left(\alpha\mu_1 + \frac{\alpha^2\sigma_1^2}{2}\right), \quad \gamma = \exp\left(\frac{\alpha^2}{2}(\sigma_2^2 - \sigma_1^2) + \alpha(\mu_2 - \mu_1)\right).$$

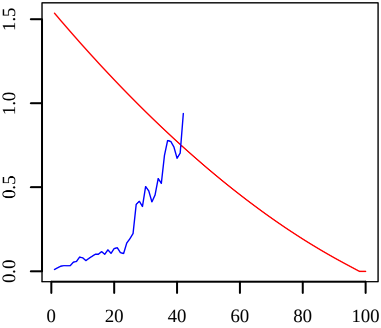
Thus, in order to find $a_\sigma(t)$ one needs to compute the functions V_t^α, starting from V_T^α, and then find their minimal non-negative roots.

A numerical example

Let $T = 100$, $\mu_1 = -\mu_2 = 1$, $\sigma_{1,2} = 1$ and θ be uniformly distributed.

The graphs below presents the solution of the problem V^0 when $\theta = 30$: the left graph — $\log S_t$; the right graph — $a_0(t)$ and ψ_t.

The optimal stopping time $\tau^* = 42$.



The stopping time τ^* can be written in therms of the posterior probability process $\pi = (\pi_t)_{t \geqslant 0}$,

$$\pi_t = \mathrm{P}(\tau \leqslant t \mid \mathscr{F}_t^X),$$

as follows:

$$\tau_\alpha^* = \inf\{0 \leqslant t \leqslant T :\ \pi_t \geqslant b_\sigma(t)\}, \qquad b_\sigma(t) = \frac{a_\sigma(t)}{1 - G(t) + a_\sigma(t)}.$$

using that

$$\pi_t = \frac{\psi_t}{1 - G(t) + \psi_t}, \qquad 0 \leqslant t \leqslant T.$$

In other words it is optimal to stop as soon as the conditional probability that the disorder has happened exceeds the level $b_\sigma(t)$.

Stopping close to θ

For given $\beta \in (0, 1)$, let $\mathfrak{M}_\beta$ denote the class of all stopping times $\tau \leqslant T$ such that $\mathrm{P}(\tau < \theta) \leqslant \beta$.

Consider the problem

$$H^\beta = \inf_{\tau \in \mathfrak{M}_\beta} \mathrm{E}(\tau - \theta \mid \tau \geqslant \theta),$$

i. e. we look for the stopping time minimizing the average delay time among all stopping times with the probability of a false alarm not greater than β.

A. N. Shiryaev (1963) proved that the optimal stopping time is

$$\tau^*_\beta = \inf\{t \geqslant 0 : \pi_t \geqslant 1 - \beta\},$$

where $\pi_t = \mathrm{P}(\theta \leqslant t \mid \mathscr{F}^X_t)$.

Applicability of these criteria

The two criteria can be represented in the form

$$\tau^* = \inf\{t \geqslant 0 : \psi_t \geqslant a(t)\},$$

where $a(t) \geqslant 0$ is a stopping boundary and the sequence ψ_t is given by

$$\psi_0 = 0, \qquad \psi_t = (p_t + \psi_{t-1}) \cdot \tfrac{\sigma^1}{\sigma^2} \exp\Big(\tfrac{(X_t-\mu_1)^2}{2\sigma_1^2} - \tfrac{(X_t-\mu_2)^2}{2\sigma_2^2}\Big).$$

Observe that

- if $X_t \approx \mu_1$ then the exponent is negative;
- if $X_t \approx \mu_2$ then the exponent is positive.

Thus, after the disorder ψ_t increases “faster” and crosses $a(t)$ “sooner”.

This fact makes it possible to expect that the criteria will work even if X_t are not i.i.d. normal random variables.

Applications to stock market data

We apply the results obtained to the problem of choosing the optimal moment of time to sell stock based on real market data.

We will use the criteria $\mathrm{E}S_\tau \to \max$ or $\mathrm{E}(\tau - \theta \mid \tau > \theta) \to \min$.

The model of choosing a moment to sell stock

1. We observe a sequence of stock prices (or index values) $S_0, S_1, \ldots, S_T$, which, we believe, has a positive trend initially.
2. It is expected that the trend will become negative by time T and

$$\log \frac{S_t}{S_{t-1}} = \begin{cases} \mu_1 + \sigma_1 \xi_t, & t < \theta \\ \mu_2 + \sigma_2 \xi_t, & t \geqslant \theta, \end{cases}$$

where $\theta \in \{t_0, \ldots, T\}$ is a random variable.

3. The parameters μ_1, σ are estimated using the data $S_0, \dots, S_{t_0}$.

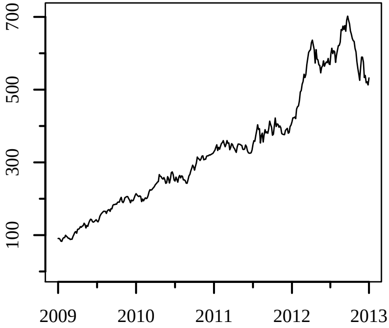
   The choice of μ_2, σ_2 and the distribution of θ is subjective. In the examples below we take $\mu_2 = -\mu_1$, $\sigma_2 = \sigma_1$ and $\theta \sim U\{t_0, \dots, T\}$, which, as we found empirically, gives good results.

   For the problem $\mathrm{E}(\tau - \theta \mid \tau > \theta) \to \min$ we take the maximum probability of a false alarm $\alpha = 0.8$.

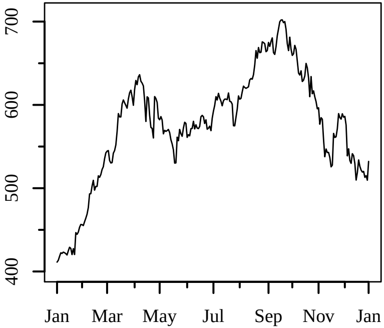
4. For a buying time $t_0 < T$ we need to find the optimal selling time τ.

1. Apple Inc, 2009-2012

During 2009-2012, Apple's stock price increased almost 9 times, from \$82.33 (6-Mar-09), to \$705.07 (21-Sep-12). By the end of 2012 it fell to \$532.17.



Apple's stock in 2009-2012



Apple's stock in 2012 .

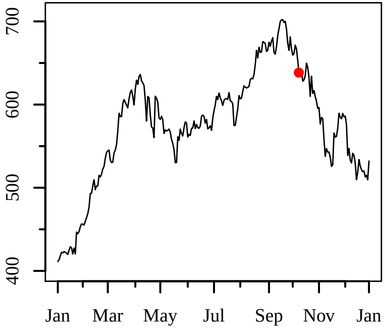
We assume $T \sim$ 31 Dec. 2012, $\mu_2 = -\mu_1$, $\sigma_2 = \sigma_1$, θ is uniformly distributed, and apply the criteria $\mathrm{E}S_\tau \to \max$.

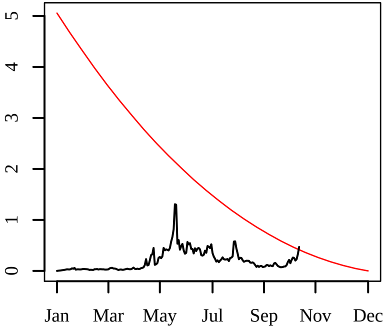
Buy	**Sell**	**% of max.**	**Return***, **%**
3-Jan-11 ($ 329.57)	9-Oct-12 ($ 635.85)	90.56	37.13
1-Jul-11 ($ 343.26)	8-Oct-12 ($ 638.17)	90.89	48.83
3-Jan-12 ($ 411.23)	8-Oct-12 ($ 638.17)	90.89	57.38
1-May-12 ($ 582.13)	9-Oct-12 ($ 635.85)	90.56	19.86
3-Jul-12 ($ 599.41)	9-Oct-12 ($ 635.85)	90.56	21.87
1-Aug-12 ($ 606.81)	11-Oct-12 ($ 628.10)	89.46	17.38

* Return = average annual return from date t_0 to date τ^*.

On the graphs – the result of applying the method when buying on January 3, 2012.

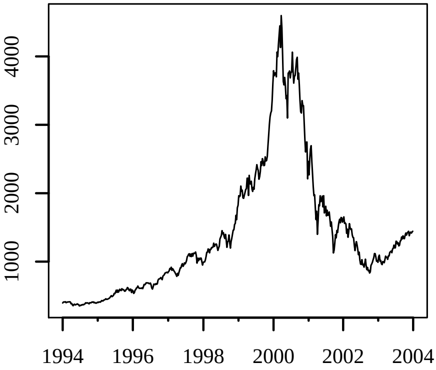
Left – the graph of the price (the red point is the selling price).
Right – the process ψ and the optimal stopping boundary.





Example 2: NASDAQ-100 during 1994–2002

From the beginning of 1994, by March 2000 the NASDAQ-100 increased more than 12 times, from 395 to 4816, and then fell to 795 by October 2002.



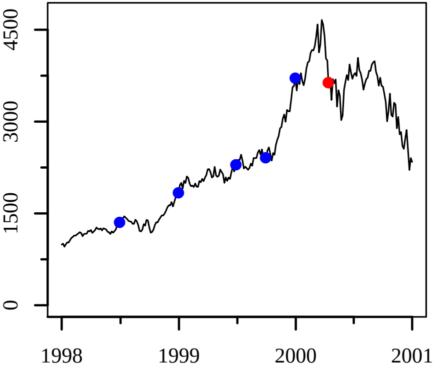
Choosing the optimal time to sell NASDAQ-100

Buy	Sell	% of max.	Return, %
2-Jul-98 ($ 1332.53)	12-Apr-00 ($ 3633.63)	77.23	56.30
4-Jan-99 ($ 1854.39)	13-Apr-00 ($ 3553.81)	75.54	50.75
1-Jul-99 ($ 2322.32)	13-Apr-00 ($ 3553.81)	75.54	53.88
1-Oct-99 ($ 2404.45)	14-Apr-00 ($ 3207.96)	68.19	53.42
3-Jan-00 ($ 3790.55)	14-Apr-00 ($ 3207.96)	68.19	-22.89

The assumptions: $T \sim$ Dec. 31, 2001; $\mu_1 = -\mu_2$, $\sigma_1 = \sigma_2$, θ is uniformly distributed.

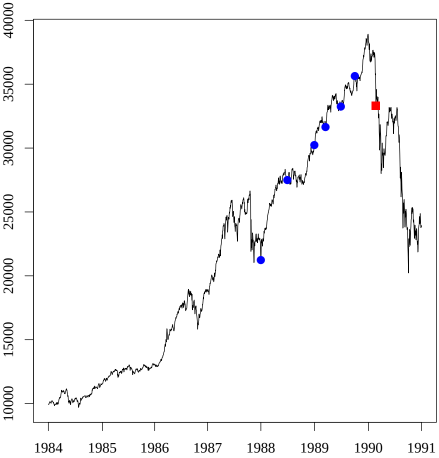
The criterion $\mathrm{E}S_\tau \to \max$.

On the graph, the buying dates are marked by the **blue** points, and April 13, 2000 (one of the selling dates) is marked by the **red** point.



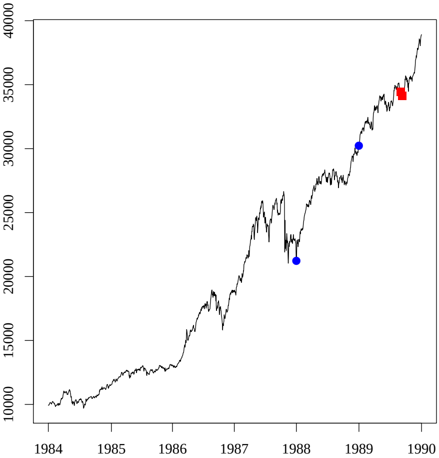
Example 3: NIKKEI-225 in the late 1980s

We applied the criterion $\mathrm{E}S_\tau \to \max$ for starting dates between 1988 and 1990, which gave the exit date 90-02-26. The assumptions on the parameters are as above; $T \sim$ the end of 1990.



Buy	Sell	% of max.	Return, %
88-01-04 (21217.00)	90-02-26 (33322.00)	85.63	21.46
88-07-01 (27504.00)	90-02-26 (33322.00)	85.63	11.85
89-01-04 (30244.00)	90-02-26 (33322.00)	85.63	8.63
89-04-03 (33042.00)	90-02-26 (33322.00)	85.63	0.95
89-07-03 (33236.00)	90-02-26 (33322.00)	85.63	0.40
89-10-02 (35623.00)	90-02-26 (33322.00)	85.63	-17.3

A good exit point is also obtained if the disorder does not happen: below is the graph when $T \sim$ the end of 1989.

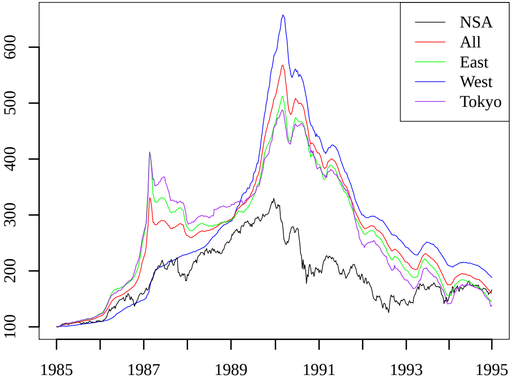


The results when the disorder does not happen are as follows.

Buy	**Sell**	**% of max.**	**Return, %**
88-01-04 (21217.00)	89-08-31 (34431.00)	88.48	29.61
88-07-01 (27504.00)	89-09-07 (34153.00)	87.76	18.50
89-01-04 (30244.00)	89-09-08 (34116.00)	87.67	17.75
89-04-03 (33042.00)	89-10-12 (34795.00)	89.41	9.79
89-07-03 (33236.00)	89-10-16 (34469.00)	88.57	12.57

Example 5: Golf course membership in Japan

The bubble of the golf course membership prices in Japan in the late 1980s was much bigger than that of NIKKEI:



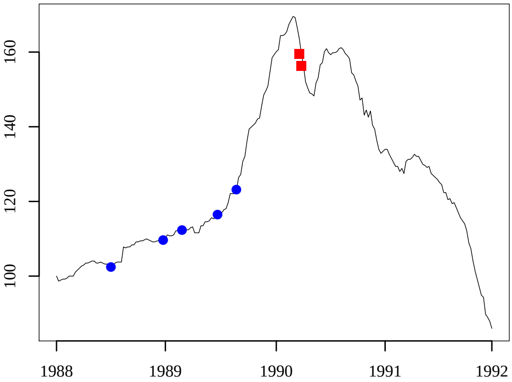
The Y-axis is % of the price at the beginning of 1985.

The results of finding the optimal selling time for the prices in Tokyo.

Buy	**Sell**	**% of max.**	**Return, %**
03.07.88 (380.00)	25.03.90 (607.00)	96.50	131.14
25.12.88 (407.00)	25.03.90 (607.00)	96.50	154.97
26.02.89 (417.00)	25.03.90 (607.00)	96.50	168.95
25.06.89 (432.00)	25.03.90 (607.00)	96.50	219.76
27.08.89 (457.00)	25.03.90 (607.00)	96.50	238.43

The assumption: $T \sim$ end of 1992, for the criterion $\mathrm{E}S_\tau \to \max$.

The buying and the selling dates on the graph:



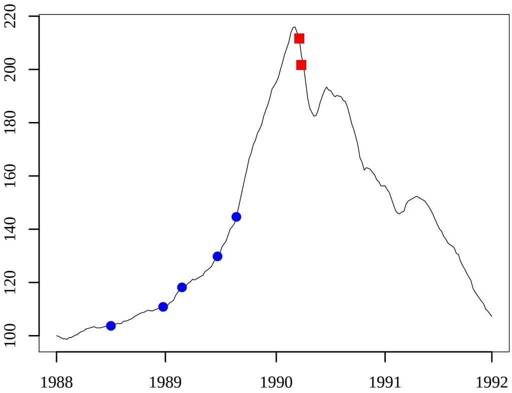
The Y-axis is % of the price at the beginning of 1988.

Applying the disorder detection rule to the nationwide prices.

Buy	**Sell**	**% of max.**	**Return, %**
03.07.88 (455.00)	25.03.90 (929.00)	98.00	199.87
25.12.88 (487.00)	25.03.90 (929.00)	98.00	250.39
26.02.89 (519.00)	25.03.90 (929.00)	98.00	261.99
25.06.89 (570.00)	25.03.90 (929.00)	98.00	315.63
27.08.89 (635.00)	25.03.90 (929.00)	98.00	319.61

The assumption: $T \sim$ end of 1992.

The buying and the selling dates on the graph:

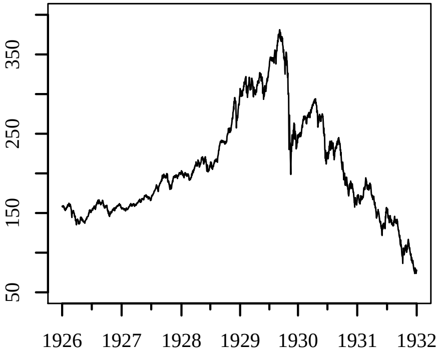


The Y-axis is % of the price at the beginning of 1988.

Example 4: The DJIA in the 1920–1930s

In the following examples we apply the criterion $\mathrm{E}(\tau - \theta \mid \tau > \theta) \to \min$ with the admissible probability of a false alarm $\alpha = 0.8$.

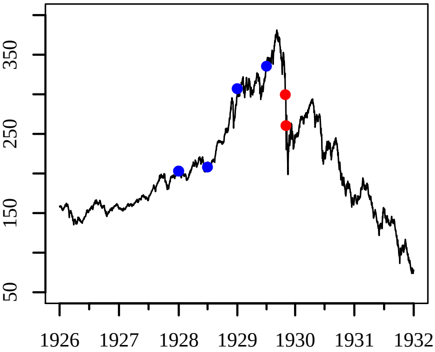
The DJIA index during 1926-1932:



Buy	Sell	% of max.	Return, %
1927-01-03 (155.16)	1931-03-19 (186.56)	48.94	3.72
1927-07-01 (168.06)	1930-06-18 (218.84)	57.41	7.60
1928-01-03 (203.35)	1929-10-28 (260.64)	68.38	11.58
1928-07-02 (208.21)	1929-10-28 (260.64)	68.38	14.33
1929-01-02 (307.01)	1929-10-24 (299.47)	78.57	-2.59
1929-07-01 (335.22)	1929-10-28 (260.64)	68.38	-64.71

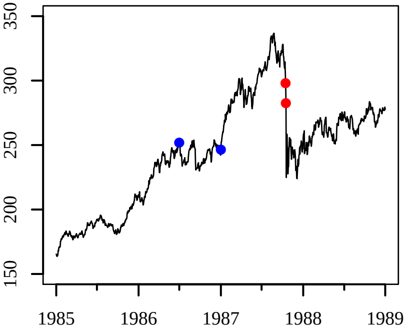
Assumptions: $T \sim$ the end of 1931, $\mu_1 = -\mu_2$, $\sigma_2 = -\sigma_1$, $\alpha = 0.8$.

The optimal selling times when buying in 1928 and 1929.



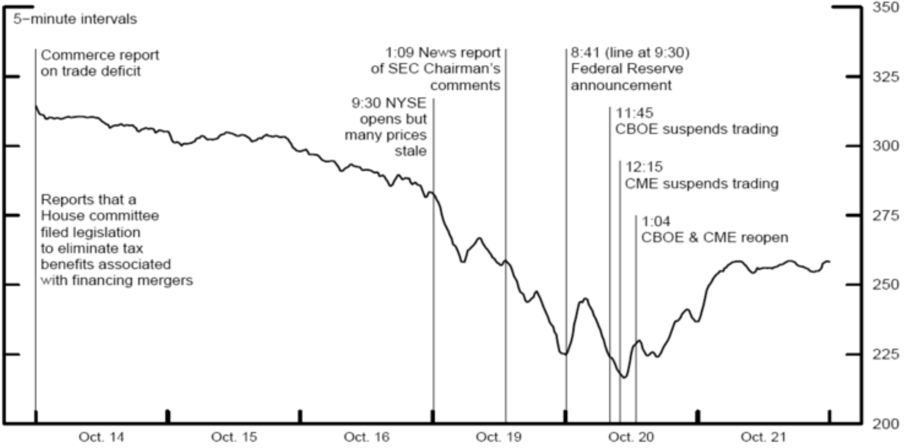
Example 6: S&P500 during the crash of 1987

The disorder detection rule suggested to close the position on Thursday or Friday before the Black Monday!



On the graph: the buying dates are 1986-07-01, and 1987-01-02; the selling dates are 1987-10-15 (Thursday) and 1987-10-15 (Friday).

S&P 500 index around the time of the crash



Source: US Federal Reserve

Applying the disorder detection rule with the parameters:
$T \sim$ the end of 1988; $\mu_2 = -\mu_1$, $\alpha = 0.8$.

Buy	**Sell**	**% of max.**	**Return, %**
1986-07-01 (252.04)	1987-10-15 (298.08)	88.51	12.93
1987-01-02 (246.45)	1987-10-16 (282.70)	83.94	17.29