Final report

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In the period September 1 to December 31, 2010 a group composed by Eulàlia Nualart, Lluís Quer-Sardanyons, Zeev Sobol and myself was hosted by the Hausdorff Research Institute for Mathematics (HIM), in the framework of a Junior Trimester Program on Stochastics.

During this period we have mostly worked on a problem of existence and regularity of the law for solutions to semilinear stochastic parabolic equations of the type

$$du(t,x) - \Delta u(t,x) + f(u(t,x)) = B\,dW(t,x), \qquad u(0,x) = u_0(x), \tag{1}$$

with $0 \leq t \leq T$, $x \in D$, where D is a bounded regular domain of $\mathbb{R}^d$, W stands for a space-time Wiener noise, $f : \mathbb{R} \to \mathbb{R}$ is an increasing function (not necessarily Lipschitz continuous) of polynomial growth, and B is a bounded linear operator on $L^2(D)$ such that the stochastic convolution

$$W_A(t) := \int_0^t S(t-s)B\,dW(s)$$

is well-defined for all $t \in [0,T]$. Here S stands for the semigroup in $L^2(D)$ generated by Δ, and W denotes the cylindrical Wiener process on $L^2(D)$ corresponding to the $d+1$-dimensional Brownian sheet in (1).

Assuming that the stochastic convolution is continuous in space and time and that $u_0 \in C^0(\overline{D})$, one has that (1) admits a unique mild solution, which is also (pathwise) continuous in space and time. Our main concern has been to investigate existence and regularity of the density of the real random variable $u(t,x)$ with respect to Lebesgue measure, for $(t,x) \in D_T :=]0,T] \times D$ arbitrary but fixed. We have used tools of the theory of (nonlinear) monotone operators and of Malliavin calculus.

To the best of our knowledge, all results related to this problem are limited to SPDE with very regular coefficients (typically f, $B \in C^1_b$). In fact, smoothness of coefficients ensures that one can write (at least formally) an equation for the Malliavin derivative $Mu(t,x)$ of $u(t,x)$ rather easily, and that one can derive from this equation enough estimates on $Mu(t,x)$ to infer that $u(t,x)$ admits a (regular) density. This scheme does not work, however, for equations with non-Lipschitz coefficients, essentially because the chain rule for the Malliavin derivative becomes inapplicable. The strategy we have adopted is to approximate the nonlinearity f in (1), thus obtaining the regularized equation

$$du_\lambda(t,x) - \Delta u_\lambda(t,x) + f_\lambda(u_\lambda(t,x)) = B\,dW(t,x),$$

where f_λ stands for the Yosida approximation of f, which is a Lipschitz continuous function converging pointwise to f as $\lambda \to 0$. One can now prove that $u_\lambda(t,x)$ is differentiable in the sense of Malliavin, and its derivative $Mu_\lambda(t,x)$ satisfies a suitable equation. We have been able to show, establishing a priori estimates for $Mu_\lambda(t,x)$ and using properties of the Malliavin derivative, to show $u(t,x)$ is Malliavin differentiable and that $Mu_\lambda(t,x)$ converges (weakly) to $Mu(t,x)$. Furthermore, passing to the limit in the equation satisfied by $Mu_\lambda(t,x)$, it has been possible to show that $Mu(t,x)$ satisfy an integral equation, from which the necessary estimates have been obtained to prove that $Mu(t,x)$ satisfy the sufficient conditions of the Bouleau-Hirsch criterion, thus concluding that $u(t,x)$ is absolutely continuous with respect to Lebesgue measure. An important ingredient in the above procedure is an equivalence result between mild and random field solutions to equation (1), which could also be of independent interest.

Further work is in progress in the following directions: (i) regularity of the density of $u(t,x)$ assuming more regularity on f (but without imposing any Lipschitz condition); (ii) extension of the results to the case of f being the sum of an increasing continuous function and a locally Lipschitz function of linear growth; (iii) multiplicative noise; (iv) more general second-order elliptic operators replacing the Laplacian.

Let me also briefly recall other research projects that have been carried out, at least in part, by members of my group during the stay at HIM:

1. E. Nualart and Ll. Quer-Sardanyons completed the work on a project they had already initiated aimed at establishing lower and upper Gaussian bounds for the probability density of the mild solution to the nonlinear stochastic heat equation in any space dimension, driven by a Gaussian noise which is white in time with some spatially homogeneous covariance. A corresponding paper is now under revision at *Stochastic Processes and their Applications.*

2. Z. Sobol has worked on a theory of SDE (on finite and, to some extent, infinite dimensional, state spaces) solvable with processes of finite lifetime. This includes a) a special topology on the space of paths with finite lifetime, which differs from the one used so far for Markov processes with finite life time; b) the corresponding theory of random processes, local martingales, and Ito processes; c) well-posedness of SDEs and martingale problems. As a corollary, he has obtained results on existence of a semigroup generated by the corresponding Kolmogorov operator and spatial Lipschitz continuity of the solution to the (backward) Kolmogorov equation.

3. Z. Sobol and I worked on the pricing of American options by an analytic approach. In particular, we characterized the value function of a general optimal stopping problem by means of a corresponding semilinear parabolic PDE with a monotone discontinuous absorption.

4. I continued my collaboration with A. Eberle (IAM, Bonn) on the rigorous study of the asymptotic properties of a class of MCMC algorithms. Part of the results will appear soon in *Probability Theory and Related Fields*.

Thanks to generous support by HIM, we organized two series of lectures, one held by Hatem Zaag (Paris XIII) on *Blow-up for the semilinear wave equation*, and a second one held by Thomas Duyckaerts (Cergy-Pontoise) on *Dynamics of the energy-critical focusing wave equation*. Moreover, the following seminar talks have been given: *Hitting probabilities and capacity for SPDEs*, by E. Nualart; *Stochastic integrals and SPDEs I & II*, by Ll. Quer-Sardanyons; *Tools for SPDEs I & II*, by myself.