REPORT ON JUNIOR TRIMESTER PROGRAM "ANALYSIS", GROUP: TOPICS IN THE THEORY OF FUNCTIONAL EQUATIONS

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1. About the research plan

Our research plan contained the following four main topic:

Functional equations on Abelian groups. In [1] Boros and Daróczy examined the following so called composite type functional equation:

$$f(x+2f(y)) = f(x)+f(y)+y\,, \qquad (x,y\in G)\,,$$

where G is an arbitrary Abelian group and $f : G \to G$. They prove that f must be a homomorphism if there is no element of order two in G. They also formulated an open problem: solve the symmetric version of the above mentioned equation, namely find all solution of equation

$$f(x+2f(y)) + f(y+2f(x)) = 2f(x)+2f(y)+x+y\,, \quad (x,y\in G)\,,$$

where G is an arbitrary Abelian group and $f : G \to G$.

Firstly, we would like to prove that f is a homomorphism also in the case, when G has element(s) of order two. On the other hand we would like to give a more detailed description such homomorphisms.

Secondly, we would like to solve the symmetric version.

Minkowski and reserved-Minkowski-type inequalities on homogeneous means, comparability of means. In [2] Daróczy made the concept of the following class of homogeneous mean values:

$$\mathcal{D}_{\alpha,p}(x,y) := \begin{cases} \left(\frac{x^p+\alpha(\sqrt{xy})^p+y^p}{\alpha+2}\right)^{1/p} & \text{if } p\neq 0\,,\ -1\le\alpha<\infty \\ \mathcal{G}(x,y) := \sqrt{xy} & \text{if } \ p=0 \text{ or } \alpha=\infty \end{cases} \qquad x,y\in\mathbb{R}_+.$$

It is a natural question to examine the comparability problem in this class, namely to find all parameters $\alpha\,,\beta\,,p\,,q$ such that $\mathcal{D}_{\alpha,p}\le\mathcal{D}_{\beta,q}$ holds. Using this inequality and the ideas of Czinder and Páles (see [4] and [3]) we can examine Minkowski and reserved-Minkowski-type inequalities in this class. More precisely, we are looking for parameters (as far as possible the best parameters) $\alpha, \beta, \gamma, p, q, r$ such that the following inequalities hold

$$\begin{aligned}\mathcal{D}_{\alpha,p}(x_1+x_2, y_1+y_2) &\leq \mathcal{D}_{\alpha,p}(x_1,y_1)+\mathcal{D}_{\alpha,p}(x_2,y_2)\,,\\ \mathcal{D}_{\alpha,p}(x_1+x_2, y_1+y_2) &\leq \mathcal{D}_{\beta,q}(x_1,y_1)+\mathcal{D}_{\gamma,r}(x_2,y_2)\end{aligned}$$

for all positive x_1, x_2, y_1, y_2.

Stability of Hermite–Hadamard inequality. The classical Hermite–Hadamard inequality, under some regularity assumptions, characterizes the convexity. That is the convex functions $f : D \to \mathbb{R}$ satisfy the so-called Hermite–Hadamard inequality

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f\big(tx+(1-t)y\big)dt \qquad (x, y \in D),$$

in other words, the integral average of the values of the function f over a segment $[x, y]$ is nonsmaller than the value of the function at the midpoint of that segment. The converse is also known to be true i.e., if a continuous f satisfies this inequality then it is also convex.

In the paper of Házy and Páles [HP08] the connection between the stability forms of the functional inequalities related to Jensen-convexity, convexity, and Hermite–Hadamard inequality when the stability term is not a constant but it depends on the closeness of the variables x and y was investigated. In other words the function $f : D \to \mathbb{R}$ satisfying

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + \delta_J(\|x-y\|),$$

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f\big(tx+(1-t)y\big)dt + \delta_H(\|x-y\|),$$

and

$$f\big(tx+(1-t)y\big) \leq tf(x)+(1-t)f(y)+\delta_C(t, \|x-y\|),$$

were considered, where $t \in [0, 1]$, $x, y \in D$, $\delta_J, \delta_H : [0, \infty[\to \mathbb{R}$, and $\delta_C : [0, 1] \times [0, \infty[\to \mathbb{R}$ are given functions called the stability terms. Our aim is to extend and generalize these results to the right-hand-side of Hermite–Hadamard inequality, that is to investigate the functions $f : D \to \mathbb{R}$ satisfying

$$\int_0^1 f\big(tx+(1-t)y\big)dt \leq \frac{f(x)+f(y)}{2} + \delta_H(\|x-y\|) \qquad (x, y \in D).$$

Stability of generalized convexity. One of the early results on regularity properties of convex functions is the so-called Bernstein–Doetsch theorem, which deduces convexity from Jensen-convexity and local upper boundedness property.

A natural extension of this theorem is to consider any kind of convexity combined with a weak regularity property and then to analyze the consequences. In this direction, where various generalizations of the Bernstein–Doetsch theorem were obtained by Nikodem and Ng [7], Házy and Páles in (see [8], [9], [10], [11], [12]).

The general aim is to obtain analogous statements for the solutions of various more general functional inequalities.

References

[1] Z. Boros and Z. Daróczy, *A composite functional equation with additive solution*, Publ. Math. Debrecen **69/1-2** (2006), 245–253.

[2] Z. Daróczy, Oral communication

[3] P. Czinder, *Inequalities on two variable Gini and Stolarsky means*, Ph.D. dissertation, **2005**, University of Debrecen.

[4] P. Czinder and Zs. Páles, *Some comparison inequalities for Gini and Stolarsky means*, Math. Inequal. Appl., accepted.

[5] A. Házy and Zs. Páles, *On a certain stability of the Hermite–Hadamard inequality* submitted to Bulletin London Math. Soc.

[6] F. Bernstein and G. Doetsch, *Zur Theorie der konvexen Funktionen*, Math. Annalen **76** (1915), 514–526.

[7] C. T. Ng and K. Nikodem, *On approximately convex functions*, Proc. Amer. Math. Soc. **118** (1993), no. 1, 103–108.

[8] Zs. Páles, *Bernstein–Doetsch-type results for general functional inequalities*, Rocznik Nauk.-Dydakt. Prace Mat. **17** (2000), 197–206, Dedicated to Professor Zenon Moszner on his 70th birthday.

[9] A. Házy and Zs. Páles, *Approximately midconvex functions*, Bulletin London Math. Soc. **36** (2004) 339–350.

[10] A. Házy, *On the stability of t-convex functions.* Aequationes Mathematicae. **74** vol 3, (2007) 210–218

[11] A. Házy, *On approximately t-convexity*, Math. Ineq. and Appl., **8** vol. 3, (2005) 389–402.

[12] A. Házy and Zs. Páles, *On approximately t-convex functions*, Publicationes Math. **66** vol. 3-4, (2005) 489–501.

2. Progress

There are two almost complete manuscripts in connection with the first and the second topic. We hadn't got time to deal with the third one. The last topic turned out to be the most successful. We wrote six publications regarding this theme:

- Pál Burai, Attila Házy and Tibor Juhász: *On approximately s-convex functions,* accepted for publication, Control and Cybernetics.
- Attila Házy, Bernstein-Doetsch-type results for h-convex functions, accepted for publication, Math. Ineq. and Appl..
- Pál Burai and Attila Házy: *On approximately h-convex functions,* Journal of Convex Analysis **18/2** (2011), 447–454.
- Pál Burai and Attila Házy: *On Orlicz-convex functions,* Proceedings of the Twelfth Symposium of Mathematics and its Applications, Editura Politechnica, Temesvár, (2010), 73–79.
- Pál Burai and Attila Házy: *Bernstein-Doetsch type results for generalized convex functions,* Proceedings of the Twelfth Symposium of Mathematics and its Applications, Editura Politechnica, Temesvár, (2010), 118–124.

- Pál Burai, Attila Házy and Tibor Juhász: *Bernstein-Doetsch type results for s-convex functions,* Publ.Math. Debrecen **75/1-2** (2009), 23–31.

3. Personal comments

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