HIM Junior Trimester Program 2008

Report on "Set-valued Numerical Analysis with Applications to Robust Optimal Control Problems"

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Abstract: Problems with "uncertainties" or "shapes" can be formulated in terms of set-valued maps in a quite general way, i.e. sets usually avoid conceptual restrictions when formulating a problem. The solution, however, is challenging – both in analysis and in numerics – since (sub-)sets do not have an obvious linear structure. Efficient numerical methods cannot omit the challenge of dimensionality.
This project has brought representatives of different branches together and it has given them an extraordinary opportunity to exchange their experience. The results cover various problems in set-valued analysis (mainly related to reachable sets of differential inclusions) and, some of them provide new numerical tools for applications in robust optimal control.

Keywords: attainable sets, differential inclusions, hybrid systems, numerical approximation schemes (generalized finite differences), robust optimal control, shadowing, viability problems for set evolutions

1 The main goals

1.1 Set-valued maps for "uncertainties" and "shapes"

Aspects of "uncertainties" and "shapes" in modeling can be handled in terms of "sets", e.g. subsets of the Euclidean space. Indeed, alternatively to probability spaces, set-valued maps permit taking more than one possibility into consideration simultaneously. Whenever shapes are described just as subsets, analytical restrictions like the regularity of boundaries can be avoided.
Subsets of $\mathbb{R}^n$, however, do not have an obvious linear structure since the Minkowski addition, for example, is not invertible or distributive. This simple observation implies challenges in both analysis and numerics.

1.2 Focus: Reachable sets – analytical investigation and numerical approximation

This project focuses on subsets of $\mathbb{R}^n$ evolving in time. Analytical investigations provide the tools for more efficient numerical methods.

In more detail, we consider a given initial value problem of ordinary differential type whose solution does not have to be unique. This gap can result from missing uniqueness on the right-hand side like in control problems, i.e. ordinary differential equations with some (a priori unknown) parameter, and, more generally, in differential inclusions. Additionally we meet lacking uniqueness of solutions whenever just some set is prescribed for the initial state (instead of a single point).
The so-called *reachable set* (a.k.a. *attainable set*) consists of *all* points in $\mathbb{R}^n$ which can be attained by some Carathéodory solution to the given initial value problem at a fixed time instant $t \geq 0$. In general it is closed and possibly bounded, but neither convex nor with smooth boundary. Hence the numerical approximation requires tools other than those for standard (smooth) ordinary differential equations.

1.3 Synthesis of methods from various (sub-) fields

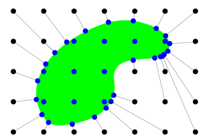
The members of this group represented several branches in set-valued and numerical analysis. Traditionally their approaches had been considered rather separately and so, the collaboration at HIM provided an extraordinary opportunity to communicate intensively and benefit from synthesis.

2 Results : A selection of (sub-)projects

2.1 Optimal control techniques for reachable set computations

Consider a nonlinear optimal control problem, i.e. first an ordinary differential equation with some "control" parameters just measurable in time and second a value functional of the solution and maybe the control. If more than one parameter is involved, but not known, it is of special interest in "robust optimal control" to investigate the value functional in the "worst case" possible. Hence the underlying reachable set is needed.

Our method in [1, 2] is based on minimizing the distances of each point in a given grid from the wanted region. It relies on so-called proximal normals similarly to the approach in [5]. The related projections require to solve optimal control problems which are handled via a direct discretization approach. These optimal control problems allow a flexible formulation and it is possible to add nonlinear state and/or control constraints and boundary conditions.



This approach will soon be applied to collision avoidance methods for cars, i.e. active steering driver assistance systems. It is continued as part of the EU Initial Training Network "Sensitivity Analysis for Deterministic Controller Design - SADCO" and in cooperation with Volkswagen AG.

2.2 Discrete approximation of impulsive differential inclusions

Technical applications like freely moving robots or unmanned aerial vehicle are related to control problems in which states can make jumps instantaneously. These hybrid systems combine aspects of time-continuous and time-discrete dynamical processes. In their special form of impulsive systems, Baier and Donchev studied the approximation of both the solution set and reachable sets by means of an Euler-like approximation scheme in [3]. Due to the potential jumps (given as a function of state), the trajectories are not continuous in general and, this implies questions about stability of the numerical scheme. Further analytical aspects of these impulsive systems are investigated in [4] – such as an exponential formula for the reachable set.

2.3 Shadowing for differential inclusions with state constraints

Differential inclusion in combination with state constraints lead to the question whether *at least* one Carathéodory solution has its values in the fixed set of constraints for all times. If not, the so-called *viability kernel* consists of all initial states with this special feature. Its numerical approximation is of increasing interest, particularly in ecological and economic models about sustainability. Rather qualitative methods (i.e. without error estimates of the Hausdorff distance) had been suggested by Saint-Pierre, Crück and collaborators before.

The aim here was to derive the first rigorous estimates for the accuracy of the fully discretized viability algorithm. A generalized form of dissipativity condition introduced by Donchev as "(relaxed) one-sided Lipschitz continuous" proved to be sufficient for estimates in unbounded time intervals and so, it laid the foundations for adapting techniques of shadowing from dynamical systems. Details were initiated at HIM, published in [9] and are part of Rieger's Ph.D. thesis [10] defended in 2009.

2.4 Differential equations for non-convex sets and extensions to control problems

State constraints are usually formulated in terms of a closed (possibly compact) subset of $\mathbb{R}^n$. In economic models, for example, the consumption of each participating agent must belong to the set of available commodities, but the latter depends explicitly on the realized consumption. Hence we need an analytical concept in which the set of constraints can evolve dynamically in time – similarly to ordinary differential equations.

Reachable sets of differential inclusions can be interpreted as a generalized form of set integration. Indeed, the special case that the right-hand side of the inclusion depends on time, but not on space, leads directly to well-known Aumann integrals, whose set values are always convex. Reachable sets overcome that restriction. This approach to set integration serves as basis for "integral equations" for compact subsets of $\mathbb{R}^n$ called *morphological equations*. Originally introduced by Aubin in the 1990s, they were extended in monograph [7] (with several sections elaborated at HIM). Taking now time-dependent parameters into consideration, we meet "control problems" for compact subsets of $\mathbb{R}^n$ – possibly even with state constraints. Several results about open-loop and closed-loop controls are presented in [8] with concrete application to image segmentation. The implementation there is based on nonlocal level set methods. Meanwhile this approach has opened new ways for dynamic *random* closed sets which are not necessarily convex [6].

2.5 Further aspects of the stay at HIM

As part of our project at HIM, a workshop on "Set-valued Numerical Analysis and Robust Optimal Control" took place at the end of March 2008. 23 experts from 10 countries participated and enjoyed the opportunity to exchange their experience in diverse branches of variational analysis.
Meanwhile four minisymposia have been prepared for the 25th IFIP TC 7 Conference on System Modeling and Optimization in Berlin in September 2011. In regard to both topics and participants, they reflect the structure of our former workshop and so, these minisymposia can be regarded as sequels.

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