Probabilistic Aspects of Rough Paths: Research Report

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Our project was centred around two related problems at the very core of the interplay between rough paths and stochastic processes. First we wished to develop a theory of rough differential equations on Lipschitz manifolds and explore applications to the study of diffusion processes on manifolds. The second problem concerned the integrability over pathspace of the Jacobian of the flow of random rough differential equations, a very well defined concrete problem, which frequently presents itself as an obstacle in the application of rough path techniques to stochastic analysis. We have fully achieved and in fact by now moved beyond the second objective and have made significant progress towards the first. Our work during the project has resulted in two papers [1] and [2] that are both available on the Arxiv.

In [1] we have (jointly with T. Lyons) developed an intrinsic coordinate free approach to rough paths on a manifold. The theory of rough paths is an extension of classical Newtonian calculus aimed at allowing models for the interaction of highly oscillatory and potentially non-differentiable systems. These systems take the form of differential equations driven by rough paths (RDEs) and naturally arise in non-linear geometric settings more general than Banach spaces. In [1] we regard rough paths as abstract objects that have integrals against sufficiently regular one forms taking their value in any Banach space. To specify the assumptions on the one form, we impose sufficient regularity on the Lipschitz manifold which we are working with. We can prove that the abstract non-linear functionals defined to be rough paths on the manifold have a notion of support that is a continuous paths on the manifold. Consequently we obtain a full characterisation of the rough paths on a Lip-γ manifold by identifying them with the pushforwards of finitely many classical rough paths from the coordinate charts. Finally, we are able to develop a notion of rough differential equations on the manifold and show the existence of solutions to such equations under suitable conditions on the vector fields.

Our entire construction is fully equivalent to the usual definition of rough paths if the manifold is a finite dimensional vector space and is the first work using rough paths in a geometric context.

In [2] we establish sharp estimates on the integrability of the Jacobian of the flow $J_{t\leftarrow 0}^{\mathbf{X}}(y_0)$ of an RDE. To understand the difficulty of this problem, we note from [5] that the standard deterministic estimate on $J_{t\leftarrow 0}^{\mathbf{X}}(y_0)$ gives

$$|J_{t\leftarrow 0}^{\mathbf{x}}(y_0)| \leq C \exp\left(C \,||\mathbf{x}||_{p\text{-var};[0,T]}^{p} \right). \tag{1}$$

But in the case where $\mathbf{X}$ is a Gaussian rough path and $p > 2$ (i.e. Brownian-type paths or rougher) the Fernique-type estimates of [4] give only that $||\mathbf{X}||_{p\text{-var};[0,T]}$ has a Gaussian tail, hence the right hand side of (1) is not integrable in general. Worse still, the work Oberhauser and Friz [3] shows that the inequality (1) can actually be saturated for a (deterministic) choice of differential equation and driving rough path. However, for random paths that have enough structure to them (in particular for Gaussian paths) only a set of small (or zero) measure comes close to equality in (1). What is therefore needed (and what we provide!) is to recast the deterministic estimate in a form that allows us to more strongly interrogate the underlying probabilistic structure

Our results in [2] allow us to deduce the existence of moments of all orders for $J_{t\leftarrow 0}^{\mathbf{x}}(y_0)$ for RDEs driven by a class of Gaussian processes (including, but not restricted to, fBm with Hurst index $H > 1/4$). In fact, our main estimate shows much more than simple moment estimates, namely that the logarithm of the Jacobian has a tail that decays faster than an exponential.

The results we obtain in [2] are relevant to a number of important problems. Firstly, they are a necessary ingredient if one wants to extend the work of [6] and [7] on the ergodicity of non-Markovian systems. Secondly, they are also an important ingredient in a Malliavin calculus proof on the smoothness of the density for RDEs driven by rough Gaussian noise in the elliptic setting. Furthermore, it allows one to achieve an analogue of Hörmander's Theorem on the smoothness of the density for Gaussian RDEs in conjunction with a suitable version of Norris's Lemma (see [10],[11]). In this context, we remark that Hu and Tindel [9] have recently obtained a Norris Lemma for fBm with $H > 1/3$ and proved smoothness-of-density results for a class of nilpotent RDEs. Hairer and Pillai [8] have also proved Hörmander-type theorems for a general class of RDEs; their results are predicated on the assumption that the Jacobian has finite moments of all order. Hence, one application of this paper is to use our tail estimate together with the results in [9] or [8] to conclude that for $t > 0$ the law of Y_t (the solution to the stochastic differential equation) will, under Hörmander's condition, have a smooth density w.r.t. Lebesgue measure on $\mathbb{R}^e$ for a rich classes of Gaussian processes X including fBm $H > 1/3$.

While we were in Bonn, we had a very successful visit by Samy Tindel from Nancy. And we intend to undertake further work which builds directly on the results we initiated during our stay at the institute.

We would like to thank Professor Kreck and his staff for providing us with such a stimulating and supportive working environment at the Hausdorff Institute. The progress we made during our stay has allowed us move beyond our original objectives and has spawned new collaborations. It is likely to have a lasting impact on the direction of our future research.

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