MIP–PDE: Solving Discrete-Continuous Nonlinear Optimal Control Problems with Linear Mixed-Integer Programming Techniques

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We applied for a three-month research visit at the Hausdorff Institute from February till April 2008. Our party consisted of four members. Martin Frank and Michael Herty brought in expertise from continuous optimization and optimal control as well as numerical techniques for solving PDE systems. Armin Fügenschuh and Lars Schewe contributed with their knowledge in the area of discrete optimization and mixed-integer linear programming. Our common research interest were modeling and solving finite-dimensional optimization problems arising from discretizations of optimal control problems with partial differential equations as constraints and additionally integer variables. These problems are typically situated in engineering applications where models for continuous phenomena are given by nonlinear partial differential equations and discrete decisions are stated by integer variables. For example, detailed dynamical models based on partial differential equations have been developed to describe gas flow on networks, where discrete decisions (opening of valves to reduce and switching of generators to increase the gas pressure) are combined with nonlinear effects due to the physics of gas. Another example with similar problems is routing of traffic flow on road networks. In order to find global optimal solutions – or at least, solutions with provable bounds on the optimality gap – one possible way is to discretize the model yielding linear mixed-integer programs. The advantage of this approach is that for the solution of the latter several highly effective numerical codes are available. However, one has to approximate the nonlinearities using only linear constraints and mixed-integer variables. Depending on the arising nonlinearities and the size of the underlying gas or traffic flow networks this leads to very large scale MILP problems, which are most likely not solvable even with today's state-of-the-art numerical solvers. Hence one has to take suitable model reductions, efficient presolve techniques or adaptive cutting planes into account that lead to reduced problems. Some of the applied reductions are most likely to be achieved by investigating the discretization of the continuous part of the model. For example, if a nonlinear constraint is discretized, the question arises which grid size is sufficient, which grid type gives the best results, and what refinement strategies are reasonable. Also, it is of interest how the discretization influences the optimal solution itself and the performance of the optimization algorithm.

In this sprit, and as our main research output at the Hausdorff Insitute, we introduced the Coolest Path Problem, which is a mixture of two well-known problems from distinct mathematical fields. One of them is the shortest path problem from combinatorial optimization. The other is the heat conduction problem from the field of partial differential equations. Together, they make up a control problem, where some geometrical object traverses a digraph in an optimal way, with constraints on intermediate or the final state. We discuss some properties of the problem and present numerical solution techniques. We demonstrate that the problem can be formulated as a linear mixed-integer program. Numerical solutions can thus be achieved within one hour for instances with up to 70 nodes in the graph. Our work on this new and exciting problem was later published in [1]. Besides our main activity, we were also able to work on other research publications, that were partly started before [2, 3], and partly involved also members from other groups being at the Institute the same time.

During our time at the Institute we organized a workshop for international PhD students and established scientists. Due to the generous funding of the Hausdorff Institute we were able to invite leading experts in the area of discrete optimization and optimal control, who are also interested in a connection of these areas: Pia Bales (TU Darmstadt), Debora Clever (TU Darmstadt), Ismael Regis des Farias (State University of New York), Kimia Gobadi (McMaster University Ontario), Simone Göttlich (TU Kaiserslautern), Martin Gugat (U Erlangen), Stephan Held (U Bonn), Christian Kirches (U Heidelberg), Oliver Kolb (TU Darmstadt), Dennis Michaels (U Magdeburg), Sebastian Sager (U Heidelberg), Marc Steinbach (U Hannover), Carsten Ziems (TU Darmstadt).

We are grateful to the Insitute for funding our work and giving us the opportunity to spend three productive months in the lovely city of Bonn. We can only recommend to our colleagues to apply for a research grant there, and send our best wishes to all persons involved in the daily work of the Insitute.

References

[1] Frank, M., Fügenschuh, A., Herty, M., Schewe, S., *The Coolest Path Problem*, Networks and Heterogeneous Media, Vol. 5, No. 1, pp. 143 162, 2010.

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