Report of Group C

Scale separation of gradient flows in continuous and discrete setting

Junior Hausdorff Trimester Program

Optimal Transportation

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The topics of the projects in our group on "scale separation of gradient flows in continuous and discrete setting" consisted in the investigation of gradient flows, their derivation and possible limits. We investigated the qualitative behavior of gradient flows in terms of the underlying metric structure. In this spirit, we showed convergence of gradient flows in some system parameters like size or numbers of particles and its long-time behavior by quantified curvature bounds.

Projects and collaborations

- Manh Hong Duong, and Julian Tugaut [DT] studied the stationary solutions of the Vlasov-Fokker-Planck (VFP) system, which is a generalization of the McKean-Vlasov equation taking in addition a friction force into account. They prove, under suitable assumptions, that the VFP equation does not have a unique stationary solution and that there exists a phase transition. This is an important preliminary step to further study the long-time behavior of this system.

- Matthias Erbar, Max Fathi, Vaios Lachos and André Schlichting collaborated in a project [EFLS] during the trimester program on the gradient flow structure of McKean-Vlasov equations on discrete spaces. They showed that a family of non-linear mean-field equations, arising in statistical mechanics, can be viewed as a gradient flow of a natural free energy functional with respect to a certain metric structure. This is a generalization of former results by [3, 4] to nonlinear equations. In addition, they prove that this

gradient flow structure arises as the limit of the gradient flow structures of a natural sequence of weakly interacting N-particle dynamics, as N goes to infinity.

- In a fellow-up work in progress, Matthias Erbar, Max Fathi, and André Schlichting investigate curvature properties of mean-field dynamics in the sense of [1]. Models in this category are mean-field Ising models like the Curie-Weiss magnet. These model show a phase transition, when the temperature crosses a critical value, which in statistical mechanics is characterized as the breakdown of convexity of the associated free energy. Preliminary results show, that the curvature changes it sign at exactly the same temperature.

- Max Fathi and Marielle Simon [FSi] present a new approach to the well-known convergence to the hydrodynamic limit for the symmetric simple exclusion process (SSEP). More precisely, they characterize any possible limit of its empirical density measures as solutions to the heat equation by passing to the limit in the gradient flow structure of the particle system.

- André Schlichting started and recently finished a project [Sch], in which he considers gradient structures for the Becker–Döring equation and its macroscopic limits. The result of Niethammer [5] is extended to proof the convergence not only for solutions of the Becker–Döring equation towards the Lifshitz–Slyozov–Wagner equation of coarsening, but also the convergence of the associated gradient structures. The gradient structure of the nonlocal coarsening equation is rigorously established. Further, it is proven that on the considered time scale the small cluster distribution of the Becker–Döring equation follows a quasistationary distribution dictated by the monomer concentration.

  The techniques of convergence of gradient flows in the projects [EFLS, FSi, Sch] is based on [6] and the involved authors took advantage in many discussions and exchange of ideas during the trimester program.

- Matthias Erbar and Nicolas Juillet [EJ] started a collaboration concerned with flows in the space of metric measure spaces during the trimester program. The project could be finalized during the follow-up workshop. They study a transformation of metric measure spaces introduced by Gigli and Mantegazza [2] consisting in replacing the original distance with the transport distance between heat kernel measures. For smooth manifolds this

flow is tangent to the Ricci flow at time zero. They study the smoothing effect of this procedure in two important examples. In the case of specific Euclidean cones, they show that a singularity persists at the apex. They generalize the construction to the Heisenberg group as an example of a sub-Riemannian manifold, and show that the space is regularized instantaneously to a smooth Riemannian manifold.

- Further projects by Max Fathi [FSh, FSt] and Emmanuel Indrei [IM, Ind] took advantage of the excellent working conditions, scientific atmosphere and fruitful discussions over the cake and coffee break.

Activities organized by the group

The research group took advantage of the winter school organized at the very beginning of the trimester program. Lecture series by Marek Biskup (UCLA), Gero Friesecke (TU München), Jan Maas (IST Austria) and Mark Peletier (TU Eindhoven) gave an introduction to fields directly or indirectly related to the overall research scheme of our group.

Thanks to the excellent organization and help by the administration of the HIM, our group was able to organize two workshops. The first had as a topic *Analytic approaches to scaling limits for random system* and offered many especially young researchers the possibility to present their results. In addition, it brought together many collaborators of participants of the trimester program to foster collaborations. Among these were in particular Richard Kraaij (TU Delft), Tony Lelièvre (Ecole des Ponts ParisTech), Michiel Renger (Weierstrass Institute), Upanshu Sharma (TU Eindhoven), Gabriel Stoltz (Ecole des Ponts ParisTech) and Julian Tugaut (Université Jean Monnet).

The second workshop on *Gradient flows and entropy methods* consisted of a lecture series by Giuseppe Savaré (Università di Pavia) and talks given by Virginie Ehrlacher (CERMICS Ecole des Ponts et Chaussées), Julian Fischer (MPI for Mathematics in the Sciences), Stefano Lisini (Università di Pavia), Daniel Matthes (TU München), Christian Seis (Universität Bonn), Yan Shu (Université Paris Ouest Nanterre La Défense), Dejan Slepčev (Carnegie Mellon University), Martin Slowik (TU Berlin), and Julian Tugaut (CNRS).

Both workshops were successful in bringing together researchers and starting fruitful discussions and collaborations.

Publications and preprints

Legend: participants and guests of the JHTP.

[DT] M.H. Duong, J. Tugaut. Stationary solutions of the Vlasov-Fokker-Planck equation: existence, characterization and phase-transition. *Applied Mathematics Letters,* 52:38–45, feb 2016.

[EFLS] M. Erbar, M. Fathi, V. Laschos, and A. Schlichting. Gradient flow structure for McKean-Vlasov equations on discrete spaces. *To appear in Discrete and Continuous Dynamical Systems - Series A,* 36(12), dec 2016. *arXiv:1601.08098.*

[EJ] M. Erbar, and N. Juillet. Smoothing and non-smoothing via a flow tangent to the Ricci flow. *arXiv:1603.00280.*

[FSh] M. Fathi, and Y. Shu. Curvature and transport inequalities for Markov chains in discrete spaces. *To appear in Bernoulli. arXiv:1509.07160.*

[FSi] M. Fathi, and M. Simon. The gradient flow approach to hydrodynamic limits for the simple exclusion process. *To appear in Particle Systems and Partial Differential Equations III, ed. A.J. Soares and P. Goncalves, Springer Proceedings in Mathematics & Statistics. arXiv:1507.06489.*

[FSt] M. Fathi, and G. Stoltz. Improving dynamical properties of stabilized discretizations of overdamped Langevin dynamics. *To appear in Numerische Mathematik. arXiv:1505.04905.*

[Ind] E. Indrei. A sharp lower bound on the polygonal isoperimetric deficit. *Proceedings of the American Mathematical Society*, 144:3115–3122, 2016

[IM] E. Indrei, and A. Minne. Nontransversal intersection of free and fixed boundaries for fully nonlinear elliptic operators in two dimensions. *Analysis & PDE*, 9(2):487–502, 2016.

[Sch] A. Schlichting Macroscopic limit of the Becker–Döring equation via gradient flows. *arXiv:1607.08735.*

References

[1] Matthias Erbar and Jan Maas. Ricci Curvature of Finite Markov Chains via Convexity of the Entropy. *Archive for Rational Mechanics and Analysis*, 206(3):997–1038, aug 2012.

[2] Nicola Gigli and Carlo Mantegazza. A flow tangent to the Ricci flow via heat kernels and mass transport. *Advances in Mathematics*, 250:74–104, 2014.

[3] Jan Maas. Gradient flows of the entropy for finite Markov chains. *Journal of Functional Analysis*, 261(8):2250–2292, oct 2011.

[4] Alexander Mielke. Geodesic convexity of the relative entropy in reversible Markov chains. *Calculus of Variations and Partial Differential Equations*, 48(1-2):1–31, sep 2013.

[5] Barbara Niethammer. On the Evolution of Large Clusters in the Becker–Döring Model. *J. Nonlinear Sci.*, 13(1):115–122, 2003.

[6] Etienne Sandier and Sylvia Serfaty. Gamma-convergence of gradient flows with applications to Ginzburg-Landau. *Comm. Pure Appl. Math.*, 57(12):1627–1672, 2004.