Report of group B

Optimal Transport and Stochastic Analysis

Junior Hausdorff Trimester Program on **Optimal transport**

January – April 2015

Goals and Group

The broad goal of our group was to push the development of (constrained) optimal transport problems with applications in stochastics with a special focus on martingale optimal transport and embedding problems. They have a natural interpretation in mathematical finance where these optimisation problems correspond to worst case scenarios. Using ideas of optimal transport in discrete time it is possible to derive general duality results which can be interpreted as superhedging results in mathematical finance. Moreover, establishing analogues of cyclical monotonicity it is often possible to explicitly construct and analyse primal optimisers which correspond to extremal models for the market in mathematical finance.

The challenge that we put ourselves for this trimester program was to extend (parts of) these results to continuous time. In parts this can be achieved by the transport approach to the Skorokhod embedding problem [BCH16b]. However, to obtain a powerful and general dual formulation as well as to provide geometrical characterizations of optimizers it is necessary to consider stochastic analysis in a pathwise manner. In particular, it is necessary to build on a powerful notion of a pathwise stochastic integral. We can subsume our goals under the following (general) problems:

Problem 1. Find a general method to calculate sharp robust bounds for option prices, incorporating market information at multiple intermediate time steps as well as historical data.

Problem 2. Develop tools that allow to analyze continuous time martingale optimal transport. In particular, develop a pathwise stochastic calculus appropriate for the challenges of martingale optimal transport.

Problem 3. Given some functional γ, find a general scheme to solve and explicitly describe (eg via PDEs) solutions to

$$\mathbb{E}[\gamma((B_s)_{s\leq\tau})] \to \max \qquad \text{subject to} \qquad B_\tau \sim \mu, \tau \in \mathfrak{C}, \tag{1}$$

where $\mathfrak{C}$ denotes a set of additional (eg distributional) contraints on τ.

We tackled these problems with a group of in total 9 people (2 of whom stayed for roughly a week, 5 for roughly a month, and 1 for two month and 1 for the whole period). Our group: Beatrice Acciaio, Mathias Beiglböck, Alex Cox, Martin Huesmann, Marcel Nutz, Harald Oberhauser, Nicolas Perkowski, David Prömel, and Pietro Siorpaes.

Results

Progress on Problem 1 (see also Progress on Problem 3):

[BCH$^+$15]: Building on and extending the duality theory for Skorokhod embedding developed in [BCH16b] together with the game theoretic approach to finance by Vladimir Vovk (cf. lecture series) M.B., A.C., M.H., N.P. and D.P. were able to prove a general multi-marginal superhedging result in continuous time.

[AL15]: During the program B.A. and Martin Larsson (invited for the workshop) made significant progress on their study of semi-static completeness in a robust framework.

Progress on Problem 2:

[Vov16]: The game-theoretic approach of Vovk provides a set-up for financial mathematics that does not require any measure theoretic foundations. A natural question to ask is if there is any link between this approach and the more commonly used 'model free' approach, where the worst case over all possible probability measures is considered. We have discussed this question with V. Vovk, albeit without clear conclusions. However, these discussions motivated Vovk to keep thinking about the problem and after the trimester he was able to give a positive answer, at least under strong assumptions.

[ŁPP16]: N.P. and D.P. have discussed game-theoretic (and thus model free) constructions of the Itô integral. These constructions are based on an approach that was developed by V. Vovk (c.f. lecture series). Previously, they had obtained a construction in the special case of continuous integrators, and in subsequent work after our departure from HIM they were able to extend this to a variety of settings allowing also for jumps, and they even strengthened the result in the case of continuous integrators.

[FP16]: During the lecture course of Peter Friz, Peter Friz and D.P. initiated an extension of the continuity of the famous Lyons-Itô map to Besov-Nikolskii topologies.

Progress on Problem 3:

[BHS15]: M.B., M.H. and Florian Stebegg (visiting for the workshop) gave a new proof of Kellerer's celebrated theorem on the existence of Markov-martingales with a continuum of prescribed marginals by establishing that the set of all martingale measures with these marginals carries a natural compact Polish topology. A key step in the proof is the observation that the martingale coupling induced by the Root embedding has a Lipschitz-Markov kernel.

[BCH16a]: M.B., A.C., and M.H. managed to extend the framework of [BCH16b] to the case of multi-marginal Skorokhod embedding, ie to solve (1) with the constraint that to each candidate stopping time τ there are $n-1$ ordered stopping times $\tau_1 \leq \ldots \leq \tau_{n-1} \leq \tau$ each embedding an a priori fixed measure μ_i. By an application of the Dambis-Dubins-Schwarz theorem this result gives also the first construction and characterisation of solutions to the multi-marginal martingale optimal transport problem. Morevoer, by an application of [BHS15], this leads to the construction of optimal PCOC's.

[ACH16]: B.A., A.C., and M.H. developed an approach to insider information/trading based on Skorokhod embedding. Mathematically, this leads to constrained embedding problems, i.e. problems of the form (1). Under some conditions on the set $\mathfrak{C}$ they are able to show a general duality theory as well as the characterisation of optimisers similar in spirit to [BCH16b]. Using the approach of [BCH$^+$15] this leads to superhedging results for the insider.

Activities

During the time at the HIM we organized

- the workshop *Optimal transport and stochastics* bringing together various researchers working on (constrained) optimal transport problems with applications in stochastic analysis, probability, statistics, and finance;
- a lecture series on *Rough Path* by Peter Friz;
- and a lecture series on *Game theoretic probability* by Vladimir Vovk; both lecture series were aiming at introducing powerful tools for pathwise stochastic calculus.

The HIM

The Hausdorff institute offered a nice and stimulating atmosphere and excellent working conditions. We are still benefitting from the stimulus of this program and looking forward to the next opportunity to spend some time at the HIM.

References

[ACH16] B. Acciaio, A M G Cox, and M Huesmann. Model-independent pricing with insider information: A Skorokhod embedding approach. *preprint*, 2016.

[AL15] Beatrice Acciaio and Martin Larsson. Semi-static completeness and robust pricing by informed investors. *arXiv preprint arXiv:1510.01890*, 2015.

[BCH+15] Mathias Beiglböck, Alexander M. G. Cox, Martin Huesmann, Nicolas Perkowski, and David Prömel. Pathwise super-replication via Vovk's outer measure. 2015.

[BCH16a] M. Beiglböck, A M G Cox, and M Huesmann. The geometry of multi-marginal Skorokhod embedding. *preprint*, 2016.

[BCH16b] Mathias Beiglböck, Alexander M. G. Cox, and Martin Huesmann. Optimal transport and Skorokhod embedding. *arXiv:1307.3656*, 2016. to appear in Invent. Math.

[BHS15] M Beiglböck, M Huesmann, and F Stebegg. Root to kellerer. *arXiv:1507.07690*, 2015.

[FP16] P. Friz and D. J. Prömel. Rough path metrics on a besov-nikolskii type scale. *arXiv:1609.03132*, 2016.

[ŁPP16] R. Łochowski, N. Perkowski, and D. J. Prömel. A superhedging approach to stochastic integration. *arXiv:1609.02349*, 2016.

[Vov16] Vladimir Vovk. Another example of duality between game-theoretic and measure-theoretic probability. *arXiv:1608.02706*, 2016.