JUNIOR HAUSDORFF TRIMESTER PROGRAM: OPTIMAL TRANSPORT (GROUP A)

JANUARY 5 - APRIL 24, 2015

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This is a report on the scientific activities of Group A (Geometry) at the Junior Hausdorff Trimester Program Optimal Transport 2015 in Bonn.

Group A studied global and local properties of non-smooth metric measure spaces. An important role plays the notion of synthetic lower Ricci curvature bounds that are defined via optimal transport techniques. This approach was initiated by celebrated works of Lott/Villani [LV09] and Sturm [Stu06a, Stu06b], and it is a non-smooth generalization of the classical notion of lower bounded Ricci tensor in smooth Riemannian geometry.

In subgroups the members started several collaborations on significant research projects, e.g. Levi-Gromov isoperimetric inequalities, existence of geometric flows for singular spaces, classification of low dimensional metric measure spaces. HIM offered possibilities to host several short time visitors who contributed with their expertise and creativity to these projects. Visitors were Fernando Galaz-Garcia (Karlsruher Institut fr Technologie), Alex Amenta (Universit Paris-Sud) and Shouhei Honda (Kyushu University).

In the course of the trimester program the group organized a workshop (New developments in Optimal Transport, Geometry and Analysis) consisting of 3 lectures courses (Nicola Gigli: Spaces with Ricci curvature bounded from below, Christina Sormani: A course on intrinsic flat convergence, Emanuel Milman: 1-D localisation) and 4 research talks (Shouhei Honda: Elliptic PDE on compact Ricci limit spaces and applications, Yashar Memarian: A Brunn-Minkowski type inequality on the sphere, Ionel Popescu: Free functional inequalities on the circle, Tapio Rajala: Tangents and dimensions of metric spaces). The workshop's aim was to present the state of the art in non-smooth geometric analysis and differential geometry for metric and metric measure spaces.

In the following we describe in more detail the scientific outcome of Group A during the Junior Trimester Program. Several collaborations have been initiated.

A. Levy-Gromov isoperimetric inequality. In [CMa] Fabio Cavalletti and Andrea Mondino develop a method for metric measure spaces to prove the celebrated Levy-Gromov isoperimetric inequality (and its generalization to arbitrary lower bounds established by E. Milman). They use the so-called 1-D localization (or needle decomposition) proposed in a recent work of B. Klartag in smooth Riemannian manifolds. This method has its roots in a work of Payne-Weinberger (1960) and was developed by Gromov-Milman and Kannan-Lovasz-Simonovits in papers of the 80-90'ies. The rough idea is to reduce the problem of establishing geometric inequalities to 1-dimensional problems. In his lecture during the workshop E. Milman gave a general introduction to 1-D localization and its applications. Cavalletti and Mondino also applied their ideas to prove sharp estimates for the p-Laplace operator and the Brunn-Minkowski inequality [CMb].

B. Rough metrics and regularity of the Gigli-Mantegazza flow. In [BLM, Ban] Lashi Bandara, Sajjad Lakzian and Michael Munn study a new geometric flow that was introduced by Gigli and Mantegazza in [GM14]. This flow is closely related to Hamilton's Ricci flow and has the special feature that it also exists for non-smooth initial geometries. Bandara, Lakzian and Munn study the Gigli-Mantegazza flow when the initial geometry is a topological manifold equipped with a Riemannian metric with measurable and bounded coefficients. This framework captures a wide class of non-smooth metric spaces and in particular the situation when the initial space has a cone-like singularity. They connect the regularity of the flow to the regularity of the initial heat kernel to study smoothening properties of the flow for explicit non-smooth inital data.

C. Variable lower Ricci curvature bounds. In [Ket] Christian Ketterer introduces a curvature-dimension condition for metric measure spaces with variable lower Ricci curvature bounds. It extends the aproach of Lott, Sturm and Villani for constant lower Ricci curvature bounds and previous work of Sturm in infinite dimensional context. A consequence is a sharp, generalized Bonnet-Myers theorem for metric measure spaces. Ketterer also proves stability and tensorization results, and compatibility with other notions of variable lower Ricci curvatures bounds.

D. Alexandrov spaces with lower Ricci curvature. In [DGGGM] Quintao Deng, Fernando Galaz-Garcia, Luis Guijarro and Michael Munn study the class of Alexandrov spaces that satisfy a positive or non-negative generalized lower Ricci curvature bound in the sense of Lott, Sturm and Villani. Alexandrov spaces are metric spaces that admit synthetic lower sectional curvature and therefore provide more regularity for the local structure. Deng, Galaz-Garcia, Guijarro and Munn were able to give a full topological classification for spaces with positive or non-negative Ricci curvature bounds in dimension 3.

E. Curvature-dimension condition in low dimensional context. In [KL] Yu Kitabeppu and Sajjad Lakzian study the class of metric measure spaces satisfying a Riemannian curvature-dimension condition $RCD^*(K,N)$ with low dimension bound N. If N is smaller than 2, they give a full classification, and in particular obtain that any such metric measure space arises as Gromov-Hausdorff limit of a sequence of Riemannian manifolds. This answers a common conjecture in low dimension. As by-product they deduce in any dimension a Bishop-Gromov-type inequality for the boundary measure of balls.

F. Geometric characterisation of L^∞-optimal transport. In [JR] Heikki Jylhä and Tapio Rajala study the L^∞-optimal transport distance in the context of general metric spaces. Though it was important in many areas of mathematics, the L^∞-transport distance is rather unexplored so far, and in contrast to its L^p-brothers little was known about its topology and its relation to weak convergence of measures. The main result of Jylhä and Rajala states that the L^∞-optimal transport distance between two probability measures can be estimated by their total cost with respect to a cost function that is a monotone function of the distance provided that the first measure has compact and connected support and the second measure is concentrated on the support of the first.

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