Junior Trimester Program "Mathematical Physics" (September - December 2012)

Final Report

Conformal field theory and Moonshine (Group A)

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The Project

Our collaboration was centered around two interrelated projects: The structure theory of vertex and generalized Kac-Moody algebras and its interplay with Mathieu moonshine.

Mathieu moonshine started with the observation of Eguchi, Ooguri and Tachikawa in 2010 that there is a weakly holomorphic mock modular form of weight 1/2, called $H(\tau)$, whose coefficients as a power series coincide with dimensions of representations of the largest Mathieu group M_{24}. This function is naturally associated to the weak Jacobi form of weight zero and index one, the universal complex elliptic genus of $K3$ surfaces. Sigma models on $K3$ surfaces possess the $\mathcal{N} = 4$ super Virasoro vertex algebra of central charge $c = 6$ as symmetry and this weak Jacobi form can be decomposed into a sum of super characters of modules for this super algebra with $H(\tau)$ counting the multiplicities. Explicit McKay-Thompson series for the conjectured M_{24}-module of Eguchi, Ooguri and Tachikawa have been proposed and then used by Gannon to establish numerically the existence of such a module. The open problem is to find a natural and mathematical rigorous defined structure on which M_{24} (or large subgroups) act by automorphisms explaining the occurrence of this M_{24}-module.

Our work during the activity at the Hausdorff Institute resulted in the two joined papers [CHM] and [CH] on Mathieu moonshine. In addition, Creutzig invited Simon Wood (IPMU Tokyo) and David Ridout (ANU Canberra) for a week in December continuing work on vertex algebras which led to the publications [CR, CRW]. Höhn started calculations related to the automorphism groups of CFTs of K3 surfaces which resulted in a joint paper with Geoffrey Mason (UC Santa Cruz) on the classification of symplectic automorphism groups associated to the second Hilbert scheme of K3 surfaces [HM].

In the note [CHM] (jointly with T. Miezaki, Yamagata University) we describe the parity of the coefficients of the McKay-Thompson series of Mathieu moonshine by using Sturm's theorem on congruences of coefficients of modular forms. As an application, we prove a conjecture of Cheng, Duncan and Harvey stated in connection with Umbral moonshine for the case of Mathieu moonshine which predicts the appearance of certain irreducible M_{24}-representations in the M_{24} moonshine module.

In the paper [CH], we relate the Mathieu group M_{24} and the complex elliptic genus of a K3 surface with the symmetries of geometric structures on K3 surfaces. Physicists have studied the sigma model on K3 surfaces resulting in a moduli space of CFTs. However, not all the postulated properties of a generic CFT from that moduli space are yet verified

mathematically rigorously. I our paper, we avoid any implicit assumptions. It is a well-known result of Mukai that a finite group of symplectic automorphisms of a $K3$ surface must be contained in a subgroup of M_{23} with at least five orbits on its natural presentation on 24 points. Our first result shows that a virtual vector space which is (in a compatible way) a representation for all groups allowed by Mukai's theorem has also the structure of a virtual M_{24} representation with the given representations of the Mukai groups as the restrictions. It follows that the complex elliptic genus of a K3 surface is a virtual module for the Mathieu group M_{24} compatible with the symplectic groups actions. Our second result shows that the topologically defined elliptic genus of a K3 surface is the graded character of a virtual module of a vertex algebra $V^{\mathrm{SU}(2)}$ called the holonomy subalgebra which contains the $\mathcal{N} = 4$ super Virasoro vertex algebra of central charge $c = 6$ as a subalgebra. We also find explicit formulas for the multiplicities of the decomposition. Finally, we prove that the equivariant topological elliptic genus for those conjugacy classes in M_{24} which arise from a symplectic action coincides with McKay-Thompson series of the proposed M_{24} moonshine module.

The orbifold second quantized elliptic genus, i.e. the orbifold elliptic genus for the symmetric powers of $K3$, coincides by a theorem of Borisov and Libgober with the elliptic genus of the corresponding Hilbert schemes. The formula for the second quantized elliptic genus formula can also be considered equivariantly similar to the formulas obtained by Borcherds in the Monstrous moonshine case. These formulas allow to compare the prediction of Mathieu moonshine with the equivariant elliptic genus of finite symplectic automorphisms of Hyperkähler manifolds deformation equivalent to a Hilbert scheme of a $K3$ surface. In particular, it is known that there are additional finite symplectic automorphisms not induced from symplectic automorphisms of a $K3$ surfaces (sometimes called non-natural automorphisms). In the paper [HM] by Höhn and Mason, all classes of finite symplectic automorphisms on Hyperkähler manifolds deformation equivalent to the second Hilbert scheme of a $K3$ surface are determined. This generalizes work by Mukai, Xiao, Kondo and Hashimoto for K3 surfaces and by Mongardi and Huybrechts for $K3^{[2]}$.

Our stay at the Hausdorff Institute

Our group invited the following speakers for the weekly Trimester Seminar: Nils Scheithauer (Darmstadt), Miranda Cheng (Paris) and David Ridout (Canberra). In addition, we invited Simon Wood (Tokyo) and Simon Norton (Cambridge, UK) for short visits in December for discussions related to our project.

We had intensive discussions with members of the group "BPS states" (Group D) which was the one most closely related to our project, but we also enjoyed interactions with the two other groups.

The trimester was filled with many activities including three workshops, three mini courses and a lecture series by Gregory Moore. During the calmer days we profited from the pleasant working atmosphere at the HIM.

The paper [HM] was motivated by discussions with Daniel Huybrechts at the beginning of the program.

References

[CHM] Gerald Höhn, Thomas Creutzig and Tsuyoshi Miezaki, *The McKay-Thompson series of Mathieu Moonshine modulo two,* The Ramanujan Journal **34** 2014, 319–328, arXiv:1211.3703.

[CH] Gerald Höhn and Thomas Creutzig, *Mathieu Moonshine and the Geometry of K3 Surfaces,* 2013, arXiv:1309.2671, accepted for Communications in Number Theory and Physics.

[CR] Thomas Creutzig and David Ridout, *Modular Data and Verlinde Formulae for Fractional Level WZW Models II,* Nucl. Phys. B **875** (2013) 423.

[CRW] Thomas Creutzig, David Ridout and Simon Wood, *Coset Constructions of Logarithmic* $(1,p)$*-Models,* Lett. Math. Phys. **104** (2014) 553–583.

[HM] Gerald Höhn and Geoffrey Mason, *Symplectic Automorphism Groups of Hyperkähler Manifolds of type* $K3^{[2]}$, 2014, in preparation.