Report on Research in Groups

Cluster Algebras and Integrable Dynamics

June 15 - August 15, 2011

Participants: Michael Gekhtman, Mikhail Z. Shapiro, Serge Tabachnikov, Alek Vainshtein

Topics

We have studied an interplay between the theory of cluster algebras and discrete integrable dynamics. Particular attention was paid to possible generalizations of the pentagram map - a discrete dynamical system, introduced by R. Schwartz in 1992 [11] whose complete integrability was proved by V. Ovsienko, R. Schwartz and S. Tabachnikov [8] and whose cluster algebra nature was observed by M. Glick [7].

Goals

Our goal was to identify a class of cluster algebras that have an “intrinsically built in” discrete integrable system and then interpret such an integrable system geometrically. The prominent roles were expected to be played by projective geometry, Poisson structures compatible with cluster algebras, and networks on surfaces.

Organization

The first week of our stay at the Hausdorff Institute was devoted to concise tutorials on background material necessary to commence the project. S. Tabachnikov lectured on the geometric and dynamical features of the pentagram map, including a description of conserved quantities and various parametrizations of the space of twisted projective polygons in which the map acts. Other participants gave presentations on various features of cluster algebras. The rest of our stay was devoted to the daily joint work on the project.

Results

We extended and generalized Glick's work by including the pentagram map into a family of discrete completely integrable systems. Our main tool is Poisson geometry of weighted directed networks on surfaces. The ingredients necessary for complete integrability – invariant Poisson brackets, integrals of motion in involution, Lax representation – are recovered from combinatorics of the networks. Postnikov [9] introduced such networks in the case of a disk and investigated their transformations and their relation to cluster transformations; most of his results are local, and hence remain valid for networks on any surface. Poisson properties of weighted directed networks in a disk and their relation to r-matrix structures on GL_n are studied in [2]. In [4] these results were further extended to networks in an annulus and r-matrix Poisson structures on matrix-valued rational functions. Applications of these techniques to the study of integrable systems can be found in [5]. A detailed presentation of the theory of weighted directed networks from a cluster algebra perspective can be found in Chapters 8–10 of [3].

Our integrable systems depend on one discrete parameter $k \geq 2$. The case $k = 3$ corresponds to the pentagram map. For $k \geq 4$, we give our integrable systems a geometric interpretation as pentagram-like maps involving deeper diagonals. If $k = 2$ and the ground field is $\mathbb{C}$, we give a geometric interpretation in terms of circle patterns [10, 1].

The results of this project have already been published in an extended research announcement [6]. A detailed version of the paper is forthcoming. The project was reported on in departmental colloquia, seminars and and international conferences including:
Bonn University, summer 2011,
Cluster Algebras and Statistical Physics, ICERM, Providence, August 2011,
BIRS Workshop, Cluster algebras, representation theory, and Poisson geometry, September 2011,
Geometry, Dynamics, Integrable Systems, Sintra, Portugal, September 2011,
Penn State University, September 2011,
Symplectic Dynamics Conference, Princeton, October 2011,
Cornell University, October 2011,
University of Illinois, December 2011,
Integrability, Modern Variations, Hausdorff Institute, January 2012 Michigan State University, February 2012,
Carolina Dynamics Symposium, April 2012,

Spring School "Tropical geometry and cluster algebras", Jussieu, Paris, April 2012,
3rd International Workshop on Combinatorics of Moduli spaces, Steklov Institute, Moscow, May 2012,
Session "Mathematical Physics and Developments in Algebra" at the 6th European Congress of Mathematics, Krakow, July 2012,
NEEDS 2012, July 2012,
Poisson 2012, July 2012,
Discrete differential geometry, Oberwolfach, July 2012,
Entretiens Jacques Cartier: Adventures in Mathematical Physics, Lyon, November 2012.

References

[1] A. Bobenko, T. Hoffmann. *Hexagonal circle patterns and integrable systems: patterns with constant angles.* Duke Math. J. **116** (2003), 525–566.

[2] M. Gekhtman, M. Shapiro, A. Vainshtein. *Poisson geometry of directed networks in a disk.* Selecta Math. **15** (2009), 61–103.

[3] M. Gekhtman, M. Shapiro, A. Vainshtein. *Cluster algebras and Poisson geometry.* Amer. Math. Soc., Providence, RI, 2010.

[4] M. Gekhtman, M. Shapiro, A. Vainshtein. *Poisson geometry of directed networks in an annulus.* J. Europ. Math. Soc., **14** (2012), 541–570.

[5] M. Gekhtman, M. Shapiro, A. Vainshtein. *Generalized Bäcklund-Darboux transformations for Coxeter-Toda flows from a cluster algebra perspective.* Acta Math. **206** (2011), 245–310.

[6] M. Gekhtman, M. Shapiro, S. Tabachnikov and A. Vainshtein, *Higher pentagram maps, weighted directed networks, and cluster dynamics*, Electron. Res. Announc. Math. Sci. **19** (2012), 1–17, DOI:10.3934/era.2012.19.1

[7] M. Glick. *The pentagram map and Y-patterns.* Adv. Math. **227** (2011), 1019–1045.

[8] V. Ovsienko, R. Schwartz, S. Tabachnikov. *The Pentagram map: a discrete integrable system.* Commun. Math. Phys. **299** (2010), 409–446.

[9] A. Postnikov. *Total positivity, Grassmannians, and networks.* Preprint math.CO/0609764.

[10] O. Schramm. *Circle patterns with the combinatorics of the square grid.* Duke Math. J. **86** (1997), 347–389.

[11] R. Schwartz. *The pentagram map.* Experiment. Math. **1** (1992), 71–81.