Report on Research in Groups

Families of automorphic forms and the trace formula

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Topics

One of the most fundamental goals in number theory is to understand automorphic representations of connected reductive groups over number fields and their most important invariants, namely their L-functions. This is the content of the Langlands program. The study of automorphic representations has far-reaching impact not only on number theory, but also on other areas.

An important concept is the notion of families of automorphic representations, with the hope that the consideration of families would enable one to understand the analytic behavior of L-functions better and attack difficult problems just as it does in geometry.

A fundamental tool to study automorphic representations is the Arthur-Selberg trace formula.

The program focused on the following topics:

- Refined spectral side of the trace formula.
- Uniform bounds of orbital integrals.
- Asymptotic properties of automorphic spectra

Goals

The objective was to find a common ground and a unifying viewpoint in which the previous results on Weyl's law, limit multiplicity formula and Sato-Tate equidistribution can be explained and extended further. Such an attempt would likely provide a guiding principle and new methods for studying automorphic families.

On the technical level we exploited several deep results, some of which need to be improved along the way. Naturally they concern Arthur's trace formula itself or the terms appearing in the spectral or geometric side of the trace formula.

1. Refined spectral side of the trace formula: The refinement due to Finis-Lapid-Müller [FLM1] (based on several previous results) is a crucial input in their work on Weyl's law and the limit multiplicity formula and should be so in the future investigation. Their inductive argument relies on two conjectural properties of intertwining operators (called properties (TWN) and (BD) in their paper), which are known only for $\mathrm{GL}(n)$ at present. A partial goal was the extension of these properties to other reductive groups.

2. Uniform bound for orbital integrals: The orbital integrals and the trace characters are the main ingredients on the geometric side of the trace formula. They are parametrized by conjugacy classes or equivalence classes of rational elements. They have a very rich structure which has been studied in great details over the years. It is crucial to have a bound for orbital integrals on p-adic groups which depends uniformly on the conjugacy classes and the prime p.

3. Quantitative version of formulas in harmonic analysis: To quantify the rate of equidistribution it would be natural to refine some well known formulas in harmonic analysis, e.g. Harish-Chandra limit formulas, Shalika germs, and trace Paley-Wiener theorems, to quantitative statements.

4. Analytic torsion: Bergeron and Venkatesh [BV] have used the analytic torsion to study the growth of torsion in the cohomology of co-compact arithmetic groups (arithmetic subgroups of reductive groups). Many arithmetic groups are not co-compact and the long-term goal is to extend the results of Bergeron and Venkatesh to the finite volume case. The main tool is again the trace formula. Its application leads to problems related to the refined spectral side mentioned above and the study of weighted orbital integrals. One focus during our time at the HIM

was on the study of the analytic torsion for congruence subgroups of GL(n).

Organization

During our stay at the HIM, T. Finis, E. Lapid, J. Matz and S. Marshall were visiting as collaborators for some of the projects. We worked mainly in pairs and had also some meetings where we discussed possible directions of study.

Results

During the activity we were able to make progress with the problems mentioned above in the following directions:

1. T. Finis and E. Lapid [FL1] extended the geometric side of Arthur's non-invariant trace formula for a reductive group to larger space of test functions which are not necessarily compactly supported. This is the space of smooth functions on the adelic group, whose derivatives of any order are absolutely integrable. It is proved that all terms on the geometric side are continues linear forms on this space of test functions.

   The main ingredients of the spectral side are logarithmic derivatives of intertwining operators. The intertwining operators can be written as the product of normalizing factors and local intertwining operators. Furthermore, the normalizing factors ca nbe expressed as quotients of automorphic L-functions. Relevant for applications of the trace formula to spectral problems as described above is the estimation of certain integrals involving the logarithmic derivative of automorphic L-functions. This is property (TWN) introduced by Finis, Lapid, and Müller in [FLM2]. It was proved in [FLM2] that this property holds for GL(n) and SL(n). Using the work of Arthur, Finis und Lapid [FL2] established (TWN) for quasi-split classical groups, for inner forms of GL(n) and SL(n), and for G_2.

2. J. Matz discussed with S. Marshall and N. Templier problems concerning the theory of spherical functions. This is related to previous work of J. Matz and N. Templier [MT], and the work of S. Marshall on

asymptotics of spherical functions. The particular problem was proving bounds for orbital integrals that would allow an improvement in the error term of asymptotic formula for the traces of Hecke operators on spaces of Maass cusp forms on $\mathrm{GL}(n)$ obtained in [MT]. They did not resolve this problem during the stay at HIM, but made significant progress, which eventually led to a full solution.

3. J. Matz and W. Müller discussed problems related to the definition of the regularized analytic torsion for quotients of the symmetric space $\widetilde{X} = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ by congruence subgroups Γ of $\mathrm{SL}(n,\mathbb{R})$. Since the corresponding locally symmetric space $\Gamma\backslash\widetilde{X}$ is not compact, the usual analytic torsion is not defined. The regularized analytic torsion is based on the definition of a regularized trace of the heat operator for the Laplace operators on forms. The definition of the regularized trace and the study of its asymptotic expansion for small time uses the Arthur trace formula. This uses the refined spectral expansion of [FLM1] and requires a detailed study of weighted orbital integrals occurring on the geometric side of the trace formula. J. Matz and W. Müller discussed the problems related to the orbital integrals during their stay at HIM, which became part of [MzM].

4. S. W. Shin and N. Templier discussed the two papers [KST1, KST2] with J.-L. Kim. We establish properties of families of automorphic representations as we vary prescribed supercuspidal representations at a given finite set of primes. For the tame supercuspidals constructed by J.-K. Yu we prove the limit multiplicity property with error terms. Thereby we obtain a Sato-Tate equidistribution for the Hecke eigenvalues of these families. We also studied quantitative aspects of trace characters Θ_π of reductive p-adic groups when the representation π varies.

5. W. Müller, S. W. Shin and N. Templier edited the Simons proceedings volume on families of automorphic forms and the trace formula, published by Springer Verlag in 2016. We also planned the 2016 Simons symposium on geometric aspects of the trace formula.

References

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