On portfolio delegation with moral hazard under translation invariance

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Overview

- Portfolio delegation as a Principal-Agent game
- Reduction to a single recursion/BSΔE
- Existence of optimal contract and strategies
- Markovian setting
- A related problem and continuous-time analog
- Further research

Context

- There is an investor (P) and a manager (A)
- (P) delegates management of her investments to (A)
- (A) has to be rewarded for his effort, which is costly
- (P) designs a contract specifying payment structure of deal
- (A) acts in his own self-interest; has an opportunity cost R
- (P) wants to maximize her own well-being
- (P) should not monitor/observe everything (A) does

Setup

- Time is discrete: $t \in \mathbb{T} := \{0, .., .T\}$
- Uncertainty is described by $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$
- The N-dim. price process $\{P_t\}$ is $\mathscr{F}_t$−adapted
- At time $t \in \mathbb{T}$, (A) and (P) have access to $\mathscr{F}_t$
- (A) has fixed initial cash endowment W_0

The financial market, wealth and cost

- (A) trades in the financial market comprising the N assets with price at time t equal P_t

$$\text{Denote } \Delta\tilde{P}_{t+1} := diag(P_t)^{-1}\Delta P_{t+1}$$

- Choosing a strategy $\{A_t\}$, the (A) generates wealth for (P):

$$\Delta W_{t+1} = A_t \cdot \Delta\tilde{P}_{t+1}$$

- Associated with each strategy $\{A_t\}$ is a cost (of effort)

$$C := \sum_t c_t(A_t)\Delta t$$

Payments and preferences

- The payment from (P) to (A) (contract) is assumed *linear*:

$$S = \varepsilon_T + \sum_{0}^{T-1} \beta_t \cdot \Delta W_{t+1}$$

where ε_T is $\mathscr{F}_T$-measurable (not just $\mathscr{F}_0$-measurable; key!)

- Terminal wealth of (P), respectively (A), is

$$W_T - S \quad \text{respectively} \quad S - \sum_{t \in \mathbb{T}} c_t(A_t) \cdot \Delta t$$

- At each time $t \in \mathbb{T}$, (A) and (P) maximize conditionally concave, translation invariant, time-consistent, ... preference functionals [2]

$$U_t^a, U_t^p : L^0(\mathscr{F}_T) \to \underline{L}^0(\mathscr{F}_t)$$

[2] See work of Cheridito, Horst or Kupper

Comments

- The Agent's optimization problems is:

  FIND AN OPTIMAL STRATEGY TAKING INTO ACCOUNT THE CONTRACTUAL AGREEMENTS.

- The Principal's problem is

  FIND AN OPTIMAL CONTRACT TAKING INTO ACCOUNT AGENT'S REACTION FUNCTION AND OUTSIDE OPTION.

- For an optimal contract it will be enough to depend on final wealth rather than its whole path [3]
- **Main idea:** make utility of (A) a control variable for (P) [4]

[3] As obtained by Ou-Yang

[4] As done by Sannikov

The Agent's problem

- (A)'s problem is a standard conditional optimization problem
- Let H_t^a utility enjoyed by/promised to (A) from $t \in \mathbb{T}$ on:

$$\begin{aligned} H_T^a &= \varepsilon_T \\ H_t^a &= \operatorname*{ess\,sup}_{A \in L^0(\mathscr{F}_t)^N} \left\{ U_t^a \left(H_{t+1}^a + \beta_t A \cdot \Delta \tilde{P}_{t+1} \right) - c_t(A) \Delta t \right\} \end{aligned}$$

- Formally if $H_{t+1}^a = \mathbb{E}\left(H_{t+1}^a | \mathscr{F}_t\right) + x_{t+1}^a$, we get a $BS\Delta E$:

$$\Delta H_{t+1}^a = x_{t+1}^a - \operatorname*{ess\,sup}_{A \in L^0(\mathscr{F}_t)^N} \left\{ U_t^a \left(x_{t+1}^a + \beta_t A \cdot \Delta \tilde{P}_{t+1} \right) - c_t(A) \Delta t \right\}$$

- Controlling ε_T steers H^a (hence x^a) and makes $H_0^a = R$. Therefore, reinterpret $BS\Delta E$ as an $S\Delta E$ starting from R

Attainability

Theorem (Existence of optimal strategies)

Assume $c_t(\cdot)$ is convex and lsc, U_t^a is seq-usc, and suppose $H_{t+1}^a < \infty$ has been found. Then, H_t^a is attained by some $A^* \in L^0(\mathscr{F}_t)$, under any of these conditions:

- *$U_t^a(\cdot)$ is sensitive to large losses and $c_t(\cdot) \geq \kappa_t + \lambda_t |\cdot|$*
- *$U_t^a(\cdot) \leq K_t + \mathbb{E}\left(\cdot | \mathscr{F}_t\right)$ and $\lim_{|A| \to \pm\infty} \frac{c_t(A)}{|A|} = \infty$*

Remark: Idea is to use No-Arbitrage arguments plus randomized Bolzano-Weierstrass

The Principal's problem

- (P)'s problem is a constrained conditional optimization one
- (P) choses strategies and contract subject to two conditions:
  - incentive compatibility (IC)
  - individual rationality (IR)
- (P)'s problem consists in solving for all time $s \in \mathbb{T}$:

$$u_s^p = \operatorname*{ess\,sup}_{\substack{\varepsilon,(\beta_k)_{k=0}^{T-1} \\ A_k \in \arg\max H_k^a \\ H_0^a \geq R}} U_s^p(W_T - S(A))$$

The first constraint is the (IC), the second the (IR) constraint.

The Principal's problem

- If A is optimal for (A), then the value of the contract is

$$\begin{aligned} S \quad = \quad & H_0^a + \sum \{ c_t(A_t)\Delta t + H_{t+1}^a - U_t^a \big(H_{t+1}^a + \beta_t A_t \cdot \Delta \tilde{P}_{t+1} \big) \\ & + \beta_t A_t \cdot \Delta \tilde{P}_{t+1} \big\} \end{aligned}$$

- (P)'s utility from $t \in \mathbb{T}$ on, under a given contract is thus:

$$U_t^p \left(\ \sum_{s \geq t} U_s^a(\ldots) - c_s(A_s)\Delta t - (\beta_s - 1) A_s \cdot \Delta \tilde{P}_{s+1} - H_{s+1}^a \ \right)$$

- (P) can steer H^a by controlling ε_T

The principal's problem

Theorem

Let h_s^p denote (P)'s optimal wealth from time s onwards. Then:

$$\begin{aligned} h_T^p &= 0 \\ h_t^p &= \operatorname*{ess\,sup}_{x \in F_{t+1}(A,\beta)} U_t^p\Big(h_{t+1}^p - x - (\beta-1)A\cdot\Delta\tilde{P}_{t+1}\Big) - c(A)\Delta t \\ &\quad + U_t^a\Big(x + \beta A\cdot\Delta\tilde{P}_{t+1}\Big) \end{aligned}$$

and $u_0^p = W_0 - R + h_0^p$

The principal's problem

- (P) controls (A)'s wealth through $x(= H^a_{t+1} - \mathbb{E}_t\left(H^a_{t+1}\right))$
- $x \in F_{t+1}(A, \beta)$ means that when faced with contract (β) and a future wealth prospect x, (A) chooses A
- Problem can be reformulated in terms of (β, A) and $\Gamma := X + \beta A \cdot \Delta\tilde{P}_{t+1} \in L^0(\mathscr{F}_{t+1})$ alone:

$$h^p_t = \operatorname*{ess\,sup}_{\substack{(A,\beta,\Gamma)\\(A,\Gamma)\in\mathbb{C}_t(\beta)}} \left\{ U^p_t\big(h^p_{t+1} + A\cdot\Delta\tilde{P}_{t+1} - \Gamma\big) - c(A)\Delta t + U^a_t(\Gamma) \right\}$$

Attainability of Principal's problem

- It is sufficient to consider an unconstrained problem:

$$V(A,\Gamma):=U_t^p\Big(h_{t+1}^p+A\cdot\Delta\tilde{P}_{t+1}-\Gamma\Big)-c(A)\Delta t+U_t^a(\Gamma)$$

- Recall standard convolution problems arising in moles of optimal risk sharing are of the form:

$$U_t^p(Y-\Gamma)+U_t^a(\Gamma)$$

  and this quantity is to be maximized over $\Gamma \in L^0(\mathscr{F}_{t+1})$
- We have complex dependencies/constraints, because

$$h_t^p = \text{ess sup}_{\{(A,\Gamma)\in\mathbb{C}_t(\beta)\}} V(A,\Gamma)$$

Attainability of Principal's problem

Theorem

- *Suppose the unrestricted problem* $\operatorname{ess\,sup}_{(A,\Gamma)} V(A,\Gamma)$ *is finite and attained at* (A^*,Γ^*)*. Then*

$$\left(1, \Gamma^*, A^*\right)$$

  forms an optimal contract and effort tuple.
- *The unconstrained problem has a solution if* $V(\cdot,\cdot) \in \underline{L}^0(\mathscr{F}_t)$*,* $U_t^{a,p}(\cdot) = \operatorname{ess\,inf}_{Z \in L_t^1(\mathscr{F}_T)}\{\alpha_t^{a,p}(Z) + \mathbb{E}[\cdot Z|\mathscr{F}_t]\}$ *plus technical conditions*

Remark: Idea is to use Komlos theorem for the Γs and random Bolzano-Weierstrass for the As

Attainability of Principal's problem

Theorem

Suppose $U_t^{a,p}(\cdot) = \frac{1}{\gamma^{a,p}} U_t(\gamma^{a,p} \cdot)$ *and that* U *satisfies conditions for attainability of Agent's problem.*

Then unconstrained problem has a solution, the optimal A^* *attains*

$$\operatorname*{ess\,sup}_A \left\{ -c(A)\Delta t + \frac{1}{\gamma} U \left(\gamma \left[h_{t+1}^p + A \Delta \tilde{P}_{t+1} \right] \right) \right\}$$

where $\gamma = \frac{\gamma^a \gamma^p}{\gamma^a + \gamma^p}$, *and* $\Gamma^* = \frac{\gamma^p}{\gamma^a + \gamma^p} \left[h_{t+1}^p + A^* \Delta \tilde{P}_{t+1} \right]$ *is optimal*[5]

[5] Inspired by a similar result by Barrieu and El-Karoui

Predictable representation property

Definition

The model has the *Predictable representation property* (PRP) if there is an adapted m-dimensional (uncorrelated) process M such that:

$$L^0(\mathscr{F}_{t+1})=\{z+Z^1\cdot\Delta P_{t+1}+Z^2\cdot\Delta M_{t+1}:z\in L^0(\mathscr{F}_t),Z^1\in[L^0(\mathscr{F}_t)]^d,Z^2\in[L^0(\mathscr{F}_t)]^M\}$$

and $R=(P,M)$ has the No-Arbitrage property.
Identify $\Gamma\in L^0(\mathscr{F}_{t+1})$, where $\mathbb{E}[\Gamma|\mathscr{F}_t]=0$, with $\gamma\in[L^0(\mathscr{F}_t)]^{d+M}$

Theorem

Suppose the PRP holds as well as the usual mild conditions on U_t^a, U_t^p and c. If the Us are sensitive to large losses then unconstrained problem is attained.

Markovian models

Suppose $\Delta P_{t+1} = P_t[\mu \Delta t + \sigma \Delta B_{t+1}]$[6]

Definition (Markovian model)

If the function $g(z) := U_t\left(z \cdot \Delta B_{t+1}\right)$ is deterministic, and PRP holds for B, we say that the problem is Markovian.

- Under suitable conditions we have the following FOCs:

$$
\begin{aligned}
0 &= [\beta\mu - \nabla c(A)]\,\Delta t + \beta\sigma\nabla g^a(\gamma) \\
0 &= [\mu - \nabla c(A)]\Delta t + \sigma\nabla g^p(\sigma' A - \gamma) \\
0 &= \nabla g^a(\gamma) - \nabla g^p(\sigma' A - \gamma)
\end{aligned}
$$

- Problem reduces to an "unconstrained" one as before and necessarily $\quad (\beta - 1)[\mu\Delta t + \sigma\nabla g^a(\gamma)] = 0$

[6] This is more related to Ou-Yang's work

Markovian models

Remark: Optimal ε is of the form $k + \sum \gamma_t \Delta B_{t+1} - \tilde{W}_T$ with k and γs constant and $\tilde{W}_T$ the optimal portfolio wealth

Remark: If further utilities have same structure, $\varepsilon = k - \frac{\gamma^a}{\gamma^a+\gamma^p}\tilde{W}_T$ and hence optimal contract has the structure [7]

$$A \mapsto S(A) = k + \frac{\gamma^p}{\gamma^a + \gamma^p} W_T^A + \frac{\gamma^a}{\gamma^a + \gamma^p}[W_T^A - \tilde{W}_T]$$

Example (Entropic utility function)

If uncertainty is spanned by Bernoulli walks and

$$U_t(X) = -\frac{1}{\gamma}\log\left(\mathbb{E}_t\left[\exp\left(-\gamma X\right)\right]\right) \text{ , then}$$

$$g(z) = -\frac{1}{\gamma}\log\left(\cosh\left(-\gamma z\sqrt{\Delta t}\right)\right)$$

[7] As in Ou-Yang

A related problem

Suppose contracts had had the form

$$S = \varepsilon + \sum [\beta_t \Delta W_{t+1} + \nu_t \Delta \tilde{P}_{t+1}]$$

with $\varepsilon \in L^0(\mathscr{F}_0)$ only.

- (A)'s utility not so readily controllable
- Translation Invariance still useful: we get two BSΔEs
- Conditional optimization problems easier than before: problem at time t involves only $\mathscr{F}_t-$measurable rvs. even without PRP
- Still $\beta_t \equiv 1$ is optimal if a related unconstrained problem is attainable

Continuous-time analog

- Define utilities through BSDEs with suitable generators:

$$U_s(X) = X + \int_s^T g_t\left(Z_t\right) dt - \int_s^T Z_t \cdot dB_t$$

- By assumption, PRP and Markovianity hold (simpler problem)
- H^a and h^p satisfy analogous BSDEs, by comparison principle
- Generalizes HJB-approach typical of these type of problems
- Non translation-invariant preferences lead to FBSDEs[8]

[8] See work of Cvitanic et al.

Comments

- Essentially, translation invariance makes BSDE-approach tractable
- With conditional analysis, we can go well beyond classical utility functions
- The price-driving process can be much general than the usual ones
- Makes no use of Maximum Principle, which in general is troublesome

To be better understood

- Explore *conditional law invariance* and relate to risk sharing and the problems above
- Study the asymmetric information case directly (through having different filtrations for (A) and (P))
- Move on to more general continuous-time models; extend conditional analysis to continuous time
- ...

Thanks