The 42 reducts of the random ordered graph

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joint work with Manuel Bodirsky and Michael Pinsker

Bonn, 2013

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Problem

Understand the reducts of homogeneous structures.

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Let Δ be homogeneous in a finite relational language.

Then the mapping

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is an anti-isomorphism from the lattice of reducts to the lattice of closed supergroups of $\mathrm{Aut}(\Delta)$.

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Conjecture (Thomas '91)

Homogeneous structures in finite relational languages have finitely many reducts.

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Moreover, any pair of homogeneous reducts $\Gamma_1 \leq \mathrm{Frlim}(\mathcal{C}_1)$, $\Gamma_2 \leq \mathrm{Frlim}(\mathcal{C}_2)$ yields a (homogeneous) reduct of $\mathrm{Frlim}(\mathcal{C})$.

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Lemma

The random ordered graph has at least 27 reducts.

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Set $T(x, y)$ iff $x < y \wedge E(x, y)$ or $x > y \wedge N(x, y)$.

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The random ordered graph has at least 27+5-1=31 reducts.

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The random ordered graph has at least 31+2=33 reducts.

Theorem (Bodirsky, Pinsker, AP ’13)

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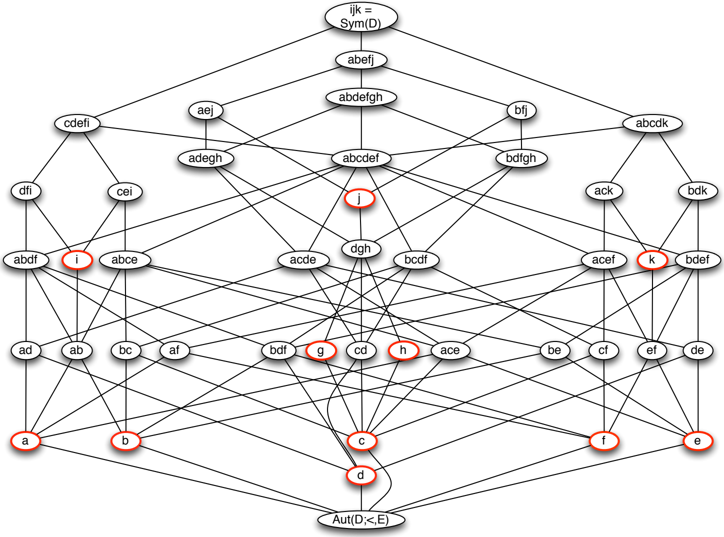

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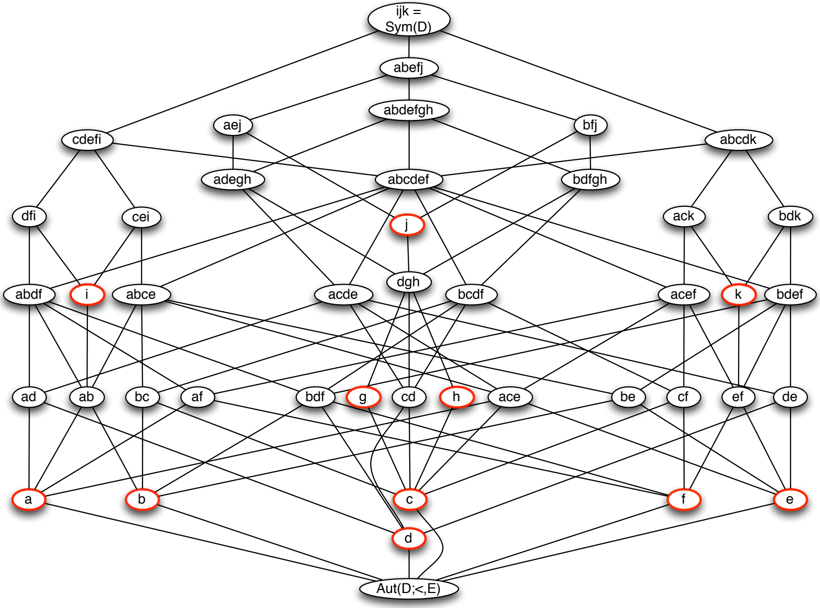
- Chacarterize the reducts of $(D; <, \prec)$.


ijk = Sym(D)
abefj
abdefgh
aej
bfj
cdefi
abcdk
adegh
abcdef
bdfgh
dfi
cei
j
ack
bdk
abdf
i
abce
acde
dgh
bcdf
acef
k
bdef
ad
ab
bc
af
bdf
g
cd
h
ace
be
cf
ef
de
a
b
c
f
e
d
Aut(D;<,E)