Infinite horizon optimization of linear stochastic systems under general time preference

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Time preference.

Time preference determines the subjective evaluation of "good" (or "bad") outcomes received at different points in time.

Definition $\mathcal{T}$ [Gafni, Torrance, 1984].

a) If $\mathbf{x}$ is a "good" then

$$\mathbf{x}\ \text{now} \begin{pmatrix} \succ \\ \sim \\ \prec \end{pmatrix} \mathbf{x}\ \text{later} \Leftrightarrow \begin{pmatrix} \text{positive} \\ \text{zero} \\ \text{negative} \end{pmatrix} \text{time preference}$$

б) If $\mathbf{x}$ is a "bad" then

$$\mathbf{x}\ \text{now} \begin{pmatrix} \succ \\ \sim \\ \prec \end{pmatrix} \mathbf{x}\ \text{later} \Leftrightarrow \begin{pmatrix} \text{negative} \\ \text{zero} \\ \text{positive} \end{pmatrix} \text{time preference}$$

Time preference.

1. We suppose that the time preference can be represented in terms of the discount function $f_t > 0$.

If time preference is $\begin{pmatrix} \text{positive} \\ \text{zero} \\ \text{negative} \end{pmatrix}$ then f_t $\begin{pmatrix} \text{decreases} \\ \text{is constant} \\ \text{increases} \end{pmatrix}$.

2. Let us define its corresponding discount rate as $\phi_t = -\dot{f}_t / f_t$.

Clearly, if time preference is $\begin{pmatrix} \text{positive} \\ \text{zero} \\ \text{negative} \end{pmatrix}$ then $\begin{pmatrix} \phi_t > 0 \\ \phi_t = 0 \\ \phi_t < 0 \end{pmatrix}$.

What do economists think of time preference?

- Conventionally trained economists support only positive time preferences that are stationary. Here the stationarity means that if the pair $(\mathbf{x}, t)$ is preferred over the pair $(\mathbf{y}, s)$, the same should also hold for $(\mathbf{x}, t+\tau)$ and $(\mathbf{y}, s+\tau)$, for any time shift $\tau > 0$. As a result, this type of preferences can be expressed only by the exponential discount function e^{-rt} $(r > 0)$.

- In many research fields of applied economics (e.g. Behavioral Economics), the concept of non-exponential positive time preference has been a subject of intensive studies, e.g. hyperbolic discount functions, see [Loewestein, Elster, 1992].

What do economists think of time preference?

- Health Economics, Economics of Renewable Resources and Energy stress the importance of zero time preference (zero discount rate), see [Olsen, 2009], [Boyle,2011], [Tyler Miller, Spoolman, 2012].

- The concept of negative time preference is widely accepted within the domain of Ecological Economics, see, e.g., [Munasinghe, 1993], [Daly, Farley, 2010].

The problem set up.

We consider a linear economic system whose evolution is described by a controlled stochastic process. Let $\tilde{X}_t$, $t \geq 0$ be an R^n–valued process defined on a complete probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$ by

$$d\tilde{X}_t = A\tilde{X}_t dt + B\tilde{U}_t dt + G dw_t\,, \qquad \tilde{X}_0 = \tilde{x}, \tag{1}$$

where $\{w_t\}_{t=0}^{\infty}$ is a d-dimensional standard Wiener process;

$\{\tilde{U}_t\}_{t=0}^{\infty}$ is an admissible control, i.e. an $\mathcal{F}_t = \sigma\{w_s,\, s \leq t\}$–adapted k-dimensional process such that there exists a solution to (1);

$\tilde{x}$ is a non-random vector;

A, B, G are non-random matrices of appropriate dimensions, $\|G\| > 0$, ($\|\cdot\|$ denotes the Euclidean matrix norm).

Let us denote by $\tilde{\mathcal{U}}$ the set of admissible controls.

The problem set up.

Consider a well-known issue of system stabilization.

Evaluation of control policies

- Let vectors $\tilde{x}_0$ and $\tilde{u}_0$ represent the target values of state and control, respectively; require that $A\tilde{x}_0 + B\tilde{u}_0 = \bar{0}$ ($\bar{0}$ is a zero vector).
- Assume that any deviation of $\tilde{X}_t$, $\tilde{U}_t$ from $\tilde{x}_0$, $\tilde{u}_0$ results in a loss.
- The loss is evaluated by economic agents according to the concept of time preference that can be expressed by means of discount function $f_t > 0$, satisfying the following

Assumption $\mathcal{D}$. The discount function $f_t > 0$, $t \geq 0$ is monotone with $f_0 = 1$, $\lim\limits_{t\to\infty} \phi_t = c_\phi$, where $\phi_t = -\dot{f}_t/f_t$ defines the corresponding discount rate, c_ϕ is a constant.

The problem set up.

The total loss over the planning horizon $[0, T]$ is measured by a quadratic cost functional J_T:

$$J_T(\tilde{U}^T) = \int_0^T f_t \left[(\tilde{X}_t' - \tilde{x}_0')Q(\tilde{X}_t - \tilde{x}_0) + (\tilde{U}_t' - \tilde{u}_0')R(\tilde{U}_t - \tilde{u}_0) \right] dt, \quad (2)$$

where $\tilde{U}^T = \{\tilde{U}_t\}_{t \le T}$ is a restriction of $\tilde{U} \in \tilde{\mathcal{U}}$ to the finite horizon $[0, T]$; Q, R are symmetric matrices, Q is positive semidefinite and R is positive definite; f_t is a discount function ($'$ denotes the matrix transpose)

We shall investigate the behavior of control system (1)-(2) as $T \to \infty$, i.e. we are to study the long-run consequences of policies.

What do mathematicians think of time preference?

- Those who works in the field of axiomatics try to account the time preference phenomenon in the most general setting possible.
- In the studies on optimal control, see, e.g., [Carlson, Leizarowitz, Haurie, 1991], discounting (usually, the exponential one) is incorporated to ensure that the expected objective function remains finite when the planning horizon tends to infinity. Negative discounting, as the authors of [Carlson, Leizarowitz, Haurie, 1991] point out, may cause destabilizing effect on the optimal trajectories. However, this feature clearly depends on the control problem, for example, in the domain of Markov Decision Processes, see [Hu, Yue, 2008], and impulsive control, see [Stettner, 1989], the use of negative discount rate is quite possible.
- Another approach to get the finite value is to compute the long-run average of objective function per unit time, i.e. the normalization proportional to the planning horizon length is applied.

Infinite time optimization and stochastic optimality.

We shall use both of the approaches: discounting is already included into the cost function and the normalization also depends on the discount function.

Referring to the infinite horizon optimization, the concept of stochastic optimality was described in [Belkina, Kabanov, Presman, 2003].

The stochastic optimality involves comparison of the cost functionals $J_T(U)$ corresponding to different controls $\tilde{U} \in \tilde{\mathcal{U}}$, in some probabilistic sense for large T's. If the criteria used there are based on the expected values of $J_T(U)$, we investigate the average optimality. Further, if some relations hold with probability 1, this could be the case for other types of optimality, e.g., almost sure optimality.

Various definitions related to the stochastic optimality will be studied in detail for the control system defined by (1)-(2).

Now let's turn to the average optimization problem set up.

The problem set up. Infinite time optimization.

We shall consider a problem of finding a control law $\tilde{U}^* \in \tilde{\mathcal{U}}$ such that

$$\limsup_{T\to\infty} \frac{EJ_T(\tilde{U}^*)}{\int\limits_0^T f_t\,dt} \le \limsup_{T\to\infty} \frac{EJ_T(\tilde{U})}{\int\limits_0^T f_t\,dt} + c_J \qquad (3)$$

holds for every $\tilde{U} \in \tilde{\mathcal{U}}$ with some constant $c_J \ge 0$.

Definition 1.

A control $\tilde{U}^* \in \tilde{\mathcal{U}}$ is called *average optimal over an infinite time horizon* if (3) holds for every $\tilde{U} \in \tilde{\mathcal{U}}$ with $c_J = 0$ and *average optimal over an infinite time horizon in the weak sense* if $c_J > 0$.

The criterion in (3) includes the expected loss per unit of cumulative discount. This kind of cost normalization (for the case of decreasing discount functions) has appeared previously in the publications on Reliability Theory, see [Van Der Weide et. al., 2008].

Average optimality over an infinite time. How to specify $\tilde{U}^*$?

Let $\tilde{U}_t^{*T}$ be an average-optimal control over a finite time interval $[0, T]$, i.e. $EJ_T(\tilde{U}^{*T}) = \inf\limits_{\tilde{U}^T \in \tilde{\mathcal{U}}^T} EJ_T(\tilde{U}^T)$. Under some conditions on the system parameters, there exists a control $\tilde{U}^* \in \tilde{\mathcal{U}}$, which can be constructed as a limit of $\tilde{U}^{*T}$ as $T \to \infty$.

These conditions include the following requirements on the cost coefficients in (2) (see [Kwakernaak, Sivan, 1972], [Belkina, Kabanov, Presman, 2003]):

a) boundedness of the matrix functions $f_t Q$, $f_t R$,

b) $f_t R$ should be bounded away from zero

Clearly, a) fails to hold for increasing f_t (negative time preference), b) is not fulfilled in the case of decreasing f_t (positive time preference). Thus, (1)-(2) is a non-standard linear quadratic regulator considered as $T \to \infty$.

Average optimality over an infinite time.

The change of variables

$$X_t := \sqrt{f_t}(\tilde{X}_t - \tilde{x}_0)\,, \quad U_t := \sqrt{f_t}(\tilde{U}_t - \tilde{u}_0). \tag{4}$$

The dynamics of the controlled process $\{X_t\}$

$$dX_t = (A - 1/2\phi_t \cdot I)X_t dt + BU_t dt + \sqrt{f_t} G dw_t\,, \qquad X_0 = \tilde{x} - \tilde{x}_0, \tag{5}$$

where I is an identity matrix, $\phi_t = -\dot{f}_t / f_t$.

The cost (2) defined on the new variables

$$J_T(U) = \int_0^T (X_t' Q X_t + U_t' R U_t)\, dt. \tag{6}$$

Note that $J_T(U) = J_T(\tilde{U})$. Further, the control system (5)-(6) has been investigated. Defining $G_t := \sqrt{f_t} G$, the relationship between the perturbation character in (5)-(6) and the discounting in the initial system can be established.

Relationship between discounting and perturbation character.

Discounting		Perturbation character	
Negative	$\phi_t < 0$	Increasing	$\|G_t\| \to \infty, t \to \infty$
Zero	$\phi_t = 0$	Constant	$\|G_t\| \equiv const$
Positive	$\phi_t > 0$	Fading	$\|G_t\| \to 0, t \to \infty$

As we have already seen, it can be shown that the substitution of variables converts (1)-(2) into a linear-quadratic regulator with a noise matrix of special form, which can be treated in an easier way, for details see [Belkina, Palamarchuk, 2012], [Palamarchuk, 2012]. Next we briefly describe the key steps.

How to study the system (5)-(6)?

The control system (5)-(6) with the noise matrix G_t

- Consider the problem $\limsup\limits_{T\to\infty}[EJ_T(U)/\int\limits_0^T \|G_t\|^2\,dt] \to \inf\limits_{U\in\mathcal{U}}$. Actually, it is a generalization of the common average cost per unit time minimization problem.
- Specify the conditions on the system parameters to guarantee the existence of optimal control U^* and prove its optimality (U^* is expected to be a linear feedback law).
- Then perform the reverse change of the variables to get the optimal control $\tilde{U}^*$ for the initial system with discounting.

The results established for the initial control problem to be presented. The conditions we are going to discuss dated back to [Belkina, Kabanov, Presman, 2003].

The specification of $\tilde{U}^*$. Assumption $\mathcal{P}$.

1. The matrix functions $A_t = A - 1/2\phi_t \cdot I$, B, Q, R, $t \geq 0$ are such that there exists the bounded absolute continuous function Π_t, $t \geq 0$ with values in the set of symmetric positive semidefinite matrices, which satisfies the Riccati equation

$$\dot{\Pi}_t + \Pi_t A_t + A_t' \Pi_t - \Pi_t B R^{-1} B' \Pi_t + Q = 0 \tag{7}$$

and such that the fundamental matrix $\Phi_{\mathcal{A}}(t,s)$ for the function $\mathcal{A}_t := A_t - BR^{-1}B'\Pi_t$ admits the exponential bound

$$\|\Phi_{\mathcal{A}}(t,s)\| \leq \kappa_1 e^{-\kappa_2 (t-s)}, \qquad s \leq t\ , \tag{8}$$

with some positive constants κ_1, $\kappa_2 > 0$.

The fundamental matrix $\Phi_{\mathcal{A}}(t,s)$ for the function $\mathcal{A}_t$ is given by

$$\frac{\partial \Phi(t,s)}{\partial t} = \mathcal{A}_t \Phi(t,s), \quad \Phi(s,s) = I \tag{9}$$

with $\Phi(t,s) = \Phi(t,0)\Phi(0,s)$, $\Phi(s,t) = \Phi^{-1}(t,s)$.

The specification of $\tilde{U}^*$. Assumption $\mathcal{P}$.

2. There is a constant $c_0 > 0$ such that for any $T > 0$ and any pair $(x_t, u_t)_{t \le T}$ satisfying

$$dx_t = A_t x_t dt + B u_t dt\,, \quad x_0 = 0\,, \tag{10}$$

the following inequality holds

$$\|x_T\|^2 + \int_0^T \|x_t\|^2\, dt \le c_0 \int_0^T (x_t' Q x_t + u_t' R u_t) dt. \tag{11}$$

Remark. Sufficient conditions to fulfill the Assumption $\mathcal{P}$ include well-known properties of linear control systems studied over an infinite time horizon such as uniformly complete controllability of the pair (A_t, B) or the exponential bound (8) when $\mathcal{A}_t = A_t$ (see also [Belkina, Kabanov, Presman, 2003], [Kwakernaak, Sivan, 1972],).

Average optimal control $\tilde{U}^*$. Theorem 1.

Suppose the Assumptions $\mathcal{P}$ and $\mathcal{D}$ hold. Then the control $\tilde{U}^*$ described by

$$\tilde{U}_t^* = -R^{-1}B'\Pi_t(\tilde{X}_t^* - \tilde{x}_0) + \tilde{u}_0\,, \tag{12}$$

a) average optimal over an infinite time horizon if in Assumption $\mathcal{D}$ the constant $c_\phi \geq 0$;
b) average optimal over an infinite time horizon in the weak sense if $c_\phi < 0\,,$

where Π_t is a bounded absolutely continuous matrix-valued function satisfying the the equation

$$\dot{\Pi}_t + \Pi_t A_t + A_t'\Pi_t - \Pi_t BR^{-1}B'\Pi_t + Q = 0\,, \tag{13}$$

$A_t = A - 1/2\phi_t \cdot I$, the process $\{\tilde{X}_t^*\}_{t=0}^{\infty}$ is given by

$$d\tilde{X}_t^* = (A - BR^{-1}B'\Pi_t)(\tilde{X}_t^* - \tilde{x}_0)dt + Gdw_t,\ \tilde{X}_0^* = \tilde{x}. \tag{14}$$

The risk of implementation $\tilde{U}^*$.

Since the control law $\tilde{U}^*$ is established, now we turn to the study of its probabilistic properties.

Definition 2.

The *deficiency process* of the control $\tilde{U}^*$ for $\tilde{U} \in \tilde{\mathcal{U}}$ is given by $\Delta_T(\tilde{U}) = J_T(\tilde{U}^*) - J_T(\tilde{U})$.

If we check all $\tilde{U} \in \tilde{\mathcal{U}}$, we get the family of $\{\Delta_T(\tilde{U})\}_{\tilde{U}\in\tilde{\mathcal{U}}}$. A risk from implementing $\tilde{U}^*$ as a long-run policy arises due to the stochastic nature of the cost $J_T(\tilde{U})$. For any $T > 0$ the cost $J_T(\tilde{U}^*)$ is a random variable, so its value may exceed the cost $J_T(\tilde{U})$ corresponding to another control $\tilde{U} \in \tilde{\mathcal{U}}$ on a particular realization. Hence, our objective is to find a deterministic function $\tilde{h}_T$ that serves as an upper bound on the difference between $J_T(\tilde{U}^*)$ and $J_T(\tilde{U})$ according to the definition below.

The risk of implementation $\tilde{U}^*$ and pathwise optimality.

Definition 3.

A non-decreasing function $\tilde{h}_T > 0$ is called an *upper function* for the family of deficiency processes $\{\Delta_T(\tilde{U})\}_{\tilde{U}\in\tilde{\mathcal{U}}}$, if for any $\tilde{U} \in \tilde{\mathcal{U}}$ there exists a finite (a.s.) random moment T_0, such that the inequlity $\Delta_T(\tilde{U}) \le \tilde{h}_T$ holds (a.s.) for all $T > T_0$.

In [Belkina, Kabanov, Presman, 2003] for the case corresponding to zero time preference the upper function $\tilde{h}_T$ was proved to have the form $\tilde{h}_T = b_0 \ln T$ ($b_0 > 0$ is some constant). It means that the difference $J_T(\tilde{U}^*) - J_T(\tilde{U})$ grows slower than the time horizon length T used in the long-run cost per unit time criterion $\limsup\limits_{T\to\infty}[J_T(U)/T]$ for optimization.

The risk of implementation $\tilde{U}^*$. The main approach.

Note that we can also investigate the problem of finding $\tilde{h}_T$ when the control system is defined by (5)-(6) with the noise matrix G_t. Let $U \in \mathcal{U}$ be fixed. In [Belkina, Kabanov, Presman, 2003] the following inequality was derived for the deficiency process Δ_T under the Assumption $\mathcal{P}$:

$$\Delta_T \leq c_1 \|X_T^*\|^2 + \mathcal{R}_T, \tag{15}$$

where $c_1, c_2 > 0$ are some constants,

$$\mathcal{R}_T := -c_2 \int_0^T \|G_t' \Pi_t x_t\|^2 \, dt - 2 \int_0^T x_t' \Pi_t G_t dw_t \,, \tag{16}$$

with (x_t, u_t) satisfying the deterministic regulator (10) and X_t^* given by

$$dX_t^* = (A_t - BR^{-1}B'\Pi_t)X_t^* dt + G_t dw_t \,, \; X_0^* = \tilde{x} - \tilde{x}_0. \tag{17}$$

Obtaining upper functions for the processes on the right side of (15), thus we get the upper function $\tilde{h}_T$ for the deficiency process Δ_T.

The upper function $\tilde{h}_T$. Theorem 2.

Let conditions of Theorem 1 hold and $\tilde{h}_T^{(0)}$ be any non-decreasing function such that $\tilde{h}_T^{(0)} \to \infty,\ T \to \infty$. Then the upper function $\tilde{h}_T$ for the family of deficiency processes $\{\Delta_T(\tilde{U})\}_{\tilde{U} \in \tilde{\mathcal{U}}}$ is given by

$$\tilde{h}_T = \max\{\tilde{h}_T^{(0)}, \tilde{h}_T^{(1)}, \tilde{h}_T^{(2)}\} \ ,$$

here $\tilde{h}_T^{(1)} = c_1 f_T \ln T$ and $\tilde{h}_T^{(2)} = c_2 f_T^{1+\beta}$,

with some constants $c_1, c_2 > 0$ and

$\beta > 0$ is an arbitrary small number.

Remark.

The results of Theorem 2 also allow to investigate almost sure (pathwise) optimality of the control $\tilde{U}^*$ according the definition given further.

The almost sure and pathwise optimality.

Definition 4.

We say that the control $\tilde{U}^* \in \tilde{\mathcal{U}}$ is *optimal almost sure (pathwise optimal)* if it is a solution to the problem

$$\limsup_{T\to\infty} \frac{J_T(\tilde{U})}{\int\limits_0^T f_t\,dt} \to \inf_{\tilde{U}\in\tilde{\mathcal{U}}} \quad \text{a.s.} \tag{18}$$

That is, for every $\tilde{U} \in \tilde{\mathcal{U}}$

$$\limsup_{T\to\infty} \frac{EJ_T(\tilde{U}^*)}{\int\limits_0^T f_t\,dt} \le \limsup_{T\to\infty} \frac{EJ_T(\tilde{U})}{\int\limits_0^T f_t\,dt} \tag{19}$$

holds a.s.

Pathwise optimality of the control $\tilde{U}^*$. Theorem 3.

Notice that the application of Theorem 2 may not be sufficient to prove the almost sure optimality of $\tilde{U}^*$ if f_t is an arbitrary function satisfying Assumption $\mathcal{D}$. However, the next statement can be established if we apply the strong law of large numbers to the pair $\left(\|X_T^*\|^2, \int\limits_0^T f_t\,dt\right)$, where the process X_t^* defined by (23), i.e. $\lim\limits_{T\to\infty}\left(\|X_T^*\|^2 / \int\limits_0^T f_t\,dt\right) = 0$ a.s.

Theorem 3.

Let the discount function f_t correspond to positive or zero time preference. If the conditions of Theorem 1 hold and $\int\limits_0^T f_t\,dt \to \infty,\ T \to \infty$, then the control law $\tilde{U}^*$ given by (12) is *almost sure (pathwise) optimal*.

Example. The Pollution Control.

Suppose that the dynamics of stock pollutant S_t is given by

$$dS_t = -aS_t dt + Z_t dt + \sigma_1 dw_t \, , \qquad S_0 = s_0, \tag{20}$$

where S_t is the stock at time t;

paramenter $a > 0$ represents the absorption capacity;

the value of $\sigma_1 > 0$ refers to impact of uncertainty on the stock dynamics;

$\{w_t\}_{t=0}^{\infty}$ is a one-dimensional standard Wiener process;

we take $\{Z_t\}_{t=0}^{\infty}$, which denotes the level of emissions at time t, as a control policy, i.e. an $\{\mathcal{F}_t\}_{t\geq 0}$-adapted process, $\mathcal{F}_t = \sigma\{w_s, \, s \leq t\}$;

s_0 – initial stock.

The Pollution control policy evaluation

We consider the cost functional over the planning horizon $[0, T]$

$$J_T(Z) = \int_0^T f_t \left(q(\bar{S} - S_t)^2 + q_1(\bar{Z} - Z_t)^2 \right) \, dt, \tag{21}$$

with some constants $q, q_1 > 0$ and $\bar{S}$, $\bar{Z}$ as the target levels of stock and emission. Note that we require $\bar{Z} = a\bar{S}$ for the stabilization purpose; the value of $\bar{S}$ could be pre-industrial stock.

As we mentioned before, environmental issues permit evaluation with respect to negative time preference. Indeed, the studies on emission timing [Schwietzke et. al., 2011], [Wigley et. al., 1996] indicate (surprisingly!) that the costs of emission control are preferable sooner rather than later.

General Hyperbolic Time preference.

Consider "hyperbolic" discount function of the form

$$f_t = 1/(1+\theta t)^{\theta_1/\theta}, \text{ where } \theta > 0, \theta_1 \text{ are constant parameters.}$$

- The value of θ indicates the degree of discount function departure from the exponential curve (as $\theta \to 0$, $f_t \to e^{\theta_1 t}$).
- The parameter θ_1 measures the decision-maker's time perception. That is, if $\theta_1 \to +\infty$, then $f_t \to 0$, time is perceived in such way that the future doesn't matter at all. If $\theta_1 > 0$, we have positive time preference, zero time preference for $\theta_1 = 0$, and $\theta_1 < 0$ leads to the case of negative time preference (slow time perception and priority of the future).

Assume $\mathcal{D}$. Then the optimal over an infinite time emission policy Z^*

$$Z_t^* = -\frac{1}{q_1}\Pi_t(S_t^* - \bar{S}) + \bar{Z}\,, \tag{22}$$

where the corresponding optimal stock $\{S_t^*\}_{t=0}^{\infty}$ is governed by

$$dS_t^* = \left(-a - \frac{\Pi_t}{q_1}\right) Z_t^* dt + \sigma_1 dw_t\,,\ S_0^* = s_0\,, \tag{23}$$

and $\Pi_t > 0$ is a bounded absolutely continuous function satisfying

$$\dot{\Pi}_t - 2a\Pi_t - \phi_t\Pi_t - \frac{\Pi_t}{q_1} + q = 0\,, \tag{24}$$

The optimal control law seems very obvious: if the stock at time t exceeds its target $\bar{S}$, the current emission level should be lowered with respect to its reference value $\bar{Z}$ and vice versa in the case of $S_t < \bar{S}$.

1. *T.A. Belkina, Yu.M. Kabanov, E.L. Presman* On a Stochastic Optimality of the Feedback Control in the LQG-Problem // Theory Probab. Appl, 2004, Vol. 48, No. 4, p. 592–603.

2. *T.A. Belkina, E.S. Palamarchuk* On stochastic optimality for a linear controller with attenuating disturbances // Autom. and Remote Control, 2013, Vol. 74, No. 4, p. 628–641.

3. *G. Boyle* Renewable Energy: Power for a Sustainable Future. 3rd ed. Oxford: OUP Oxford, 2012.

4. *D.A. Carlson,A. Haurie, A. Leizarowitz* Infinite horizon optimal control: deterministic and stochastic systems. Berlin: Springer-Verlag, 1991.

5. *H.E. Daly, J. Farley* Ecological Economics: Principles and Applications. 2nd ed. Washington D.C., 2010.

References.

6. *A. Gafni, G.W. Torrance* Risk attitude and time preference in health // Management Science, 1984, Vol. 30, No. 4, p. 440–451.

7. *Q. Hu, W. Yue* Markov Decision Processes With Their Applications. N.-Y.: Springer, 2008.

8. *H. Kwakernaak, R. Sivan* Linear Optimal Control Systems. N.-Y.: Wiley-Interscience, 1972.

9. *G. Loewestein, J. Elster (eds.)* Choice over time. N.-Y.: Russell Sage Foundation, 1992.

10. *M. Munasinghe* Environmental economics and sustainable development. Washington D.C.: World Bank Publications, 1993.

11. *J.A. Olsen* Principles in Health Economics and Policy. Oxford: Oxford University Press, 2009.

12. *E.S. Palamarchuk* Risk estimation for linear economic systems under negative time preferences // Economics and Mathematical Methods (in Russian), 2013, Vol. 49, No. 3, p. 113–130.

13. *S. Schwietzke, W.M. Griffin, H.S. Matthews* Relevance of emissions timing in biofuel greenhouse gases and climate impacts // Environ. Sci. Technol., 2011, Vol. 45, No. 19, p. 8197-–8203.

14. *L. Stettner* On Some Stopping and Impulsive Control Problems with a General Discount Rate Criteria // Probability and Mathematical Statistics, 1989, Vol. 10, Fasc. 2, p. 223–245.

References.

15. *G. Tyler Miller, Jr., Scott E. Spoolman* Living in the Environment: Principles, Connections, and Solutions. 17th ed. Belmont: Brooks/Cole, 2012.

16. *T.M.L. Wigley, R. Richels, J.A. Edmonds* Economic and environmental choices in the stabilization of atmospheric CO2 concentrations // Nature, 1996, Vol. 378, No. 6562, p. 240–243.

16. *J.A.M. Van Der Weide, J.M. Van Noortwijk, Suyono* Renewal Theory with Discounting in Advances In Mathematical Modeling For Reliability. Amsterdam: IOS Press, 2008.