Metric homogeneous structures

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I. Background

A Polish group is a topological group whose topology is induced by a complete separable metric.

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Example

The permutation group of the integers, denoted by $\mathcal{S}_\infty$.
If $\sigma, \tau \in \mathcal{S}_\infty$, let $d(\sigma,\tau) = \inf\{2^{-n} \colon \sigma_{\restriction n} = \tau_{\restriction n}\}$.
This metric is left-invariant but not complete; the following metric is complete, but not left-invariant:

$$d'(\sigma,\tau) = d(\sigma,\tau) + d(\sigma^{-1},\tau^{-1}).$$

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Any closed subgroup of $\mathcal{S}_\infty$ is a Polish group.

Observation

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Theorem (Folklore)

Let G be a closed subgroup of $\mathcal{S}_\infty$. Then there exists a homogeneous countable relational structure $\mathcal{M}$ such that G is isomorphic (as a topological group) to $\mathrm{Aut}(\mathcal{M})$.

"Homogeneous" means that any isomorphism between finitely-generated substructures extends to an automorphism of the whole structure.

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Definition

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- each P_i is a K_i-Lipschitz map from some M^{n_i}, endowed with the sup-metric, to $\mathbb{R}$.

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$\mathcal{M}$ is Polish if (M, d) is a Polish metric space.

Examples of Polish metric structures

Any first-order countable relational structure (endowed with the discrete metric), any Polish metric space.
More interesting examples: a separable Hilbert space, the measure algebra of a standard probability space, the Urysohn space, the Gurarij space...

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It is interesting to try and transfer known concepts and techniques of the model theory of countable structures to the context of Polish metric structures.

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Now "homogeneous " means that, if $\bar{a}$, $\bar{b}$ are such that the substructures of $\mathcal{M}$ generated by $\bar{a}$, $\bar{b}$ are isomorphic via an isomorphism sending $\bar{a}$ to $\bar{b}$, then for any $\varepsilon > 0$ there exists $g \in \mathrm{Aut}(\mathcal{M})$ such that $d(g(\bar{a}), \bar{b}) < \varepsilon$.

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Examples: the Urysohn space, the separable Hilbert space, the Gurarij space (where ε's are really needed)...

II. Ample generics

Definition (Hodges–Hodkinson–Lascar–Shelah)

Let a Polish group G act on itself by conjugacy, and then let G act on G^n by the diagonal product of this action.
Then G has ample generics if for any n there is a comeager orbit in G^n.

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Theorem (Kechris–Rosendal)

Let G be a Polish group with ample generics, and H a separable topological group. Then any homomorphism from G to H is necessarily continuous.

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Question

Does there exist a Polish group which has ample generics and is not isomorphic to a closed subgroup of $\mathcal{S}_\infty$?

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Definition

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- (G, τ) is a Polish group.
- ∂ is a bi-invariant metric on G that refines τ.
- Each set $\{(g, h) \colon \partial(g, h) \leq r\}$ is closed in G^2. (i.e. ∂ is τ-lower semicontinuous)

(Under these conditions, ∂ must be complete, though that will not play a major role in this talk)

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- The isometry group of any separable Banach space, endowed with the topology of pointwise convergence and the metric induced by the operator norm. Of particular interest to us: the isometry group of a separable Hilbert space.
- The automorphism group of a standard probability space (X, μ), endowed with
  - The weak topology, induced by the maps $g \mapsto \mu(g(A) \Delta A)$;
  - The uniform metric, defined e.g by the formula

$$\partial(g, h) = \mu(\{x \colon g(x) \neq h(x)\}).$$

Observation

Given a Polish group (G, τ), there exists a coarsest uniformity generated by bi-invariant entourages which refines the left uniformity on G; this uniformity is induced by a bi-invariant metric ∂_u and (G, τ, ∂_u) is a Polish topometric group.

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If d is a left-invariant distance compatible with the topology of G, then one can set

$$\partial_u(g, h) = \sup_{k \in G} d(gk, hk) \ .$$

Definition

Let (G, τ, ∂) be a Polish topometric group. Define

$$(A)_{<\varepsilon} = \{g \in G \colon \exists a \in A \ \partial(g, a) < \varepsilon\}$$

Analysis will usually enable us only to speak of $(A)_{<\varepsilon}$ for arbitrarily small $\varepsilon > 0$ - controlling $\overline{A}^{\partial}$ rather than A.

Definition (Ben Yaacov–Berenstein–M.)

Let (G, τ, ∂) be a Polish topometric group, and let G act on G^n by diagonal conjugacy (denoted by $*$).
G has ample metric generics if for any n and any $\varepsilon > 0$ there exists $\bar{g} \in G^n$ such that $(G * \bar{g})_{<\varepsilon}$ is comeagre.

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Definition

A Polish group (G, τ) has ample metric generics if (G, τ, ∂_u) has ample generics.

Examples (Ben Yaacov–Berenstein–M.)

The unitary group of a separable Hilbert space, the automorphism group of a standard atomless probability space and the isometry group of the Urysohn space have ample metric generics (and conjugacy classes are known to be meager in these groups).

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Theorem (Ben Yaacov–Berenstein-M.)

Let (G, τ, ∂) be a Polish topometric group with ample metric generics, and H be a second-countable topological group. Assume that $\varphi\colon G \to H$ is a homomorphism that is continuous from (G, ∂) to H. Then φ is continuous from (G, τ) to H.

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Philosophy: the existence of ample metric generics is used to reduce a question about τ to a formally easier question about ∂_u.

The existence of groups having ample metric generics without ample generics is somewhat counterintuitive: the existence of a dense diagonal conjugacy class in G^2, and the bi-invariance of ∂_u, imply that ∂_u is constant outside of a meager set. So one extends a meager set by taking its closure for an almost discrete metric, and one ends up with a comeager set...

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We would then like to understand better how the metric and topology interact.

Question

Assume (G, τ, ∂) is a Polish topometric group, and A is comeager in some open set O. Must $(A)_{<r}$ be comeager in $(O)_{<r}$ for $r > 0$?

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Question

Does there exist a topometric version of Effros' theorem characterizing elements with a comeager orbit under some continuous action of a Polish group?

III. Gray sets

(from now on everything is joint work with Itaï Ben Yaacov)

Convention: In the rest of this talk (X, τ, ∂) always stands for a Polish topometric space; this means that the following conditions are satisfied:

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- ∂ refines τ and is τ-lower semicontinuous.

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When moving to ample generics in the context of Polish topometric groups, what really matters in not the element g but the distance function $\partial(g, \cdot)$.

Definition

A gray subset of X is a function from X to $[0, +\infty]$.

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Examples

- If $A \subseteq X$, its zero-indicator function $\mathbf{0}_A$ is defined by

$$\mathbf{0}_A(x) = \begin{cases} 0 & \text{if } x \in A \\ +\infty & \text{if } x \notin A \end{cases}$$

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- $\partial \colon X^2 \to [0, +\infty]$ is a gray subset of X^2.

We write $A \sqsubseteq X$ to mean that A is a gray subset of X; $A \sqsubseteq B$ means that $A(x) \geq B(x)$ for all x, accordingly $A \sqcup B$ denotes $\min(A, B)$, $A \sqcap B$ stands for $\max(A, B)$.

Definition

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When applied to zero-indicator functions, this yields the usual definitions of open/closed sets. The axiom stating that ∂ is τ-lower semicontinuous may then be viewed as a separation axiom (stating that the diagonal is closed).

Definition

For $A \sqsubseteq X$, the interior A° of A is the union of all gray subset of A - equivalently, this is the inf of all u.s.c functions greater than A, and one has the formula

$$\forall x \in X \quad A^\circ(x) = \limsup_{y \to x} A(y) \ .$$

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One can define similarly the closure of a gray set.

Definition

$A \sqsubseteq X$ is meager if there exists $r > 0$ such that $\{x\colon A(x) \leq r\}$ is meager; A is comeager if $\{x\colon A(x) \leq r\}$ is comeager for all $r > 0$, equivalently, if $\{x\colon A(x) = 0\}$ is comeager.

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Definition

We write $A \sqsubseteq^* B$ if $\{x\colon B(x) \leq A(x)\}$ is comeager. Then, as in usual descriptive set theory, one can define

$$U(A) = \sqcup\{U \sqsubseteq^o X\colon U \sqsubseteq^* A\}$$

It is always the case that $U(A) \sqsubseteq^* A$.

Then, one wants to say that $A \sqsubseteq X$ is Baire-measurable if $A =^* U(A)$, equivalently, if $A \sqsubseteq^* U(A)$.

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One can go about developing an analogue of descriptive set theory in this context - defining G_δ gray sets, relative topological notions, stating (and proving) versions of the Baire category and Kuratowski–Ulam theorems for gray sets...

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In particular, if a subset A of X is comeager in an open subset O, then $(A)_{<r}$ is comeager in $(O)_{<r}$ for all r.

Corollary

Assume that X satisfies the above assumption, and A is 1-Lipschitz. Then $U(A)$ is also 1-Lipschitz.

Theorem (Effros)

Let G be a Polish group acting continuously on a Polish space X. Then $x \in X$ is generic iff $G \cdot x$ is dense and the map $g \mapsto g \cdot x$ is open (from G to $G \cdot x$).

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What is the topometric extension of this theorem?

Theorem

Let G be a Polish group acting continuously on a Polish topometric space (X, τ, ∂). Assume that $(U)_{<r}$ is open for any open U and $r > 0$.
Then $x \in X$ is a metric generic iff $G \cdot x$ is dense and the map $g \mapsto g \cdot x$ is metrically open (from G to $G \cdot x$), i.e. :
For any $\varepsilon > 0$, for any open $U \subseteq G$, $(U \cdot x)_{<\varepsilon}$ is open in $G \cdot x$.
When this holds, $\overline{G \cdot x}^{\partial}$ must be G_δ.

IV. Gray subgroups

From now on G is a topological group whose topology is induced by a complete metric. We are used to multiplying subsets of G:

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One can also set $A^{-1} = \{a^{-1} \colon a \in A\}$.

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Definition

Given $A, B \sqsubseteq G$, define

$$A * B(g) = \inf_{xy=g} A(x) + B(y) \ .$$

Also, $A^{-1}(g) = A(g^{-1})$.

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Proposition ("Pettis lemma")

For any $A, B \sqsubseteq G$, $U(A) * U(B) \sqsubseteq A * B$.
In particular, if A is Baire-measurable and nonmeager, then $(A * A^{-1})^{\circ}(1) = 0$.

A subset H of G is a subgroup iff H is nonempty and $H \cdot H^{-1} \subseteq H$.
For gray subsets, the corresponding conditions become $\inf(H) = 0$ and $H * H^{-1} \sqsubseteq H$; this is the same as:

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- $H(1) = 0$.
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So subgroups correspond to seminorms or, equivalently, to left-invariant pseudometrics on G (via the correspondence $H(\cdot) \leftrightarrow d(1, \cdot)$).

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Observation

Let $\mathcal{M}$ be an $\aleph_0$-categorical Polish structure, and $H \sqsubseteq Aut(\mathcal{M})$ be an open real-valued gray subgroup. Then there exists an imaginary $a \in M^{eq}$ such that $H(g) = d(a, g(a))$.

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Definition

A Polish topometric group (G, τ, ∂) has the small density property if any ∂-Baire measurable left-invariant pseudometric of density character $< 2^{\aleph_0}$ is open.

What does the small index property become in the topometric context? Subgroups become left-invariant pseudometrics; index then should translate to density character of the corresponding quotient space.

Definition

A Polish topometric group (G, τ, ∂) has the small density property if any ∂-Baire measurable left-invariant pseudometric of density character $< 2^{\aleph_0}$ is open.

The Baire-measurability above simply comes from the fact that we should impose some mild definability assumption - otherwise the uniform metric ∂ plays no part in the definition.

Theorem (Ben Yaacov–M.)

Any Polish topometric group with ample generics has the small density property.

Theorem (Ben Yaacov–M.)

Any Polish topometric group with ample generics has the small density property.

This is essentially a reformulation of the automatic continuity theorem we saw earlier in the talk - but the approach via gray sets makes the proof more transparent.

Thank you for your attention!