Metric homogeneous structures

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I. Background
A **Polish group** is a topological group whose topology is induced by a complete separable metric.
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**Example**

The permutation group of the integers, denoted by $S_\infty$. If $\sigma, \tau \in S_\infty$, let $d(\sigma, \tau) = \inf\{2^{-n}: \sigma|_n = \tau|_n\}$. This metric is left-invariant but not complete; the following metric is complete, but not left-invariant:

$$d'(\sigma, \tau) = d(\sigma, \tau) + d(\sigma^{-1}, \tau^{-1}).$$
A Polish group is a topological group whose topology is induced by a complete separable metric.

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Any closed subgroup of $S_\infty$ is a Polish group.
Observation
The automorphism group of a first-order countable structure is a closed subgroup of \( S_\infty \): identify an automorphism with the permutation that it induces on the universe of the structure.
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Theorem (Folklore)
Let $G$ be a closed subgroup of $S_\infty$. Then there exists a homogeneous countable relational structure $M$ such that $G$ is isomorphic (as a topological group) to $\text{Aut}(M)$.

“Homogeneous” means that any isomorphism between finitely-generated substructures extends to an automorphism of the whole structure.
Observation
Not every Polish group is (topologically, or even abstractly) isomorphic to a subgroup of $S_\infty$. 

J. Melleray  
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Definition
In this talk, a metric structure $\mathcal{M}$ is a family $(M, d, (P_i)_{i \in I}, (K_i)_{i \in I}, (f_j)_{j \in J}, (L_j)_{j \in J})$ where:

- $(M, d)$ is a complete metric space.
- Each $P_i$ is a $K_i$-Lipschitz map from some $M_{n_i}$, endowed with the sup-metric, to $\mathbb{R}$.
- Each $f_j$ is an $L_j$-Lipschitz map from some $M_{m_j}$ to $M$.

$M$ is Polish if $(M, d)$ is a Polish metric space.
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Any first-order countable relational structure (endowed with the discrete metric), any Polish metric space. More interesting examples: a separable Hilbert space, the measure algebra of a standard probability space, the Urysohn space, the Gurarij space...
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It is interesting to try and transfer known concepts and techniques of the model theory of countable structures to the context of Polish metric structures.
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- If $\mathcal{M}$ is a Polish metric structure, its automorphism group $\text{Aut}(\mathcal{M})$, endowed with the pointwise convergence topology, is a Polish group.
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- Conversely, for any Polish group $G$ there exists a homogeneous relational Polish metric structure $\mathcal{M}$ such that $G$ is isomorphic (as a topological group) to $\text{Aut}(\mathcal{M})$. 

Examples: the Urysohn space, the separable Hilbert space, the Gurarij space (where $\varepsilon$'s are really needed)...
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• If $\mathcal{M}$ is a Polish metric structure, its automorphism group $\text{Aut}(\mathcal{M})$, endowed with the pointwise convergence topology, is a Polish group.

• Conversely, for any Polish group $G$ there exists a \textbf{homogeneous} relational Polish metric structure $\mathcal{M}$ such that $G$ is isomorphic (as a topological group) to $\text{Aut}(\mathcal{M})$.

Now “homogeneous ” means that, if $\bar{a}$, $\bar{b}$ are such that the substructures of $\mathcal{M}$ generated by $\bar{a}$, $\bar{b}$ are isomorphic via an isomorphism sending $\bar{a}$ to $\bar{b}$, then for any $\varepsilon > 0$ there exists $g \in \text{Aut}(\mathcal{M})$ such that $d(g(\bar{a}), \bar{b}) < \varepsilon$. 
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II. Ample generics
Definition (Hodges–Hodkinson–Lascar–Shelah)
Let a Polish group $G$ act on itself by conjugacy, and then let $G$ act on $G^n$ by the diagonal product of this action. Then $G$ has **ample generics** if for any $n$ there is a comeager orbit in $G^n$. 

Theorem (Kechris–Rosendal)
Let $G$ be a Polish group with ample generics, and $H$ a separable topological group. Then any homomorphism from $G$ to $H$ is necessarily continuous.

Question
Does there exist a Polish group which has ample generics and is not isomorphic to a closed subgroup of $S_\infty$?
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**Definition**

A Polish topometric group is a triple \((G, \tau, \partial)\) such that

1. \((G, \tau)\) is a Polish group.
2. \(\partial\) is a bi-invariant metric on \(G\) that refines \(\tau\).
3. Each set \(\{ (g, h) : \partial(g, h) \leq r \}\) is closed in \(G^2\). (i.e. \(\partial\) is \(\tau\)-lower semicontinuous)

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- Any Polish group, endowed with the discrete metric.
- The isometry group of any separable Banach space, endowed with the topology of pointwise convergence and the metric induced by the operator norm. Of particular interest to us: the isometry group of a separable Hilbert space.
- The automorphism group of a standard probability space $(X, \mu)$, endowed with
  - The **weak topology**, induced by the maps
    \[ g \mapsto \mu(g(A) \Delta A); \]
  - The **uniform metric**, defined e.g. by the formula
    \[ \partial(g, h) = \mu(\{ x : g(x) \neq h(x) \}). \]
Observation
Given a Polish group \((G, \tau)\), there exists a coarsest uniformity generated by bi-invariant entourages which refines the left uniformity on \(G\); this uniformity is induced by a bi-invariant metric \(\partial_u\) and \((G, \tau, \partial_u)\) is a Polish topometric group.
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If \(d\) is a left-invariant distance compatible with the topology of \(G\), then one can set

\[
\partial_u(g, h) = \sup_{k \in G} d(gk, hk).
\]
Definition
Let \((G, \tau, \partial)\) be a Polish topometric group. Define

\[(A)_{<\varepsilon} = \{ g \in G : \exists a \in A \: \partial(g, a) < \varepsilon \}\]

Analysis will usually enable us only to speak of \((A)_{<\varepsilon}\) for arbitrarily small \(\varepsilon > 0\) - controlling \(\overline{A}^\partial\) rather than \(A\).
Definition (Ben Yaacov–Berenstein–M.)

Let \((G, \tau, \partial)\) be a Polish topometric group, and let \(G\) act on \(G^n\) by diagonal conjugacy (denoted by \(\ast\)). \(G\) has **ample metric generics** if for any \(n\) and any \(\varepsilon > 0\) there exists \(\bar{g} \in G^n\) such that \(\langle G \ast \bar{g} \rangle_{<\varepsilon}\) is comeagre.

Equivalently: for any \(n\) there exists (a comeagre set of) \(\bar{g} \in G^n\) such that \(G \ast \bar{g} \partial\) is \(\tau\)-comeagre.

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A Polish group \((G, \tau)\) has ample metric generics if \((G, \tau, \partial_u)\) has ample generics.
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**Definition**

A Polish group \((G, \tau)\) has **ample metric generics** if \((G, \tau, \partial_u)\) has ample generics.
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The unitary group of a separable Hilbert space, the automorphism group of a standard atomless probability space and the isometry group of the Urysohn space have ample metric generics (and conjugacy classes are known to be meager in these groups).
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Theorem (Ben Yaacov–Berenstein-M.)

Let \((G, \tau, \partial)\) be a Polish topometric group with ample metric generics, and \(H\) be a second-countable topological group. Assume that \(\varphi: G \to H\) is a homomorphism that is continuous from \((G, \partial)\) to \(H\). Then \(\varphi\) is continuous from \((G, \tau)\) to \(H\).
Theorem (Ben Yaacov–Berenstein-M)
Let \((X, \mu)\) be a standard atomless probability space. Then \(\text{Aut}(X, \mu)\) has the automatic continuity property.

Theorem (Tsankov)
Let \(H\) be a separable Hilbert space. Then the unitary group \(U(H)\) has the automatic continuity property. (This uses our approach via the uniform metric but additional work is required)

Philosophy: the existence of ample metric generics is used to reduce a question about \(\tau\) to a formally easier question about \(\partial u\).
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Philosophy: the existence of ample metric generics is used to reduce a question about $\tau$ to a formally easier question about $\partial_u$.
The existence of groups having ample metric generics without ample generics is somewhat counterintuitive: the existence of a dense diagonal conjugacy class in $G^2$, and the bi-invariance of $\partial_u$, imply that $\partial_u$ is constant outside of a meager set. So one extends a meager set by taking its closure for an almost discrete metric, and one ends up with a comeager set...
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We would then like to understand better how the metric and topology interact.
Question
Assume \((G, \tau, \partial)\) is a Polish topometric group, and \(A\) is comeager in some open set \(O\). Must \((A)_<r\) be comeager in \((O)_<r\) for \(r > 0\)?
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Assume \((G, \tau, \partial)\) is a Polish topometric group, and \(A\) is comeager in some open set \(O\). Must \((A)_r\) be comeager in \((O)_r\) for \(r > 0\)?

Question
Does there exist a topometric version of Effros’ theorem characterizing elements with a comeager orbit under some continuous action of a Polish group?
III. Gray sets
(from now on everything is joint work with Itaï Ben Yaacov)
Convention: In the rest of this talk \((X, \tau, \partial)\) always stands for a Polish topometric space; this means that the following conditions are satisfied:

- \((X, \tau)\) is Polish.
- \(\partial\) refines \(\tau\) and is \(\tau\)-lower semicontinuous.
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When moving to ample generics in the context of Polish topometric groups, what really matters in not the element \(g\) but the distance function \(\partial(g, \cdot)\).
Definition
A gray subset of $X$ is a function from $X$ to $[0, +\infty]$. 
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Examples

• If $A \subseteq X$, its zero-indicator function $0_A$ is defined by

$$0_A(x) = \begin{cases} 0 & \text{if } x \in A \\ +\infty & \text{if } x \notin A \end{cases}$$
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- $\partial : X^2 \rightarrow [0, +\infty]$ is a gray subset of $X^2$.

We write $A \sqsubseteq X$ to mean that $A$ is a gray subset of $X$; $A \sqsubseteq B$ means that $A(x) \geq B(x)$ for all $x$, accordingly $A \sqcup B$ denotes $\min(A, B)$, $A \sqcap B$ stands for $\max(A, B)$. 

J. Melleray  
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Definition

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When applied to zero-indicator functions, this yields the usual definitions of open/closed sets. The axiom stating that $\partial$ is $\tau$-lower semicontinuous may then be viewed as a separation axiom (stating that the diagonal is closed).
Definition

For $A \subseteq X$, the interior $A^\circ$ of $A$ is the union of all gray subset of $A$ - equivalently, this is the inf of all u.s.c functions greater than $A$, and one has the formula

$$\forall x \in X \quad A^\circ(x) = \limsup_{y \to x} A(y).$$
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$$\forall x \in X \quad A^\circ(x) = \limsup_{y \to x} A(y).$$

One can define similarly the closure of a gray set.
Definition

$A \subseteq X$ is meager if there exists $r > 0$ such that $\{x : A(x) \leq r\}$ is meager; $A$ is comeager if $\{x : A(x) \leq r\}$ is comeager for all $r > 0$, equivalently, if $\{x : A(x) = 0\}$ is comeager.
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Definition

We write $A \sqsubseteq^* B$ if $\{x : B(x) \leq A(x)\}$ is comeager. Then, as in usual descriptive set theory, one can define

$$U(A) = \bigsqcup \{U \sqsubseteq^o X : U \sqsubseteq^* A\}$$

It is always the case that $U(A) \sqsubseteq^* A$. 
Then, one wants to say that $A \subseteq X$ is Baire-measurable if $A =^* U(A)$, equivalently, if $A \subseteq^* U(A)$.
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Of course, since we are dealing with functions from $X$ to $[0, +\infty]$, there is a natural notion of Baire-measurability; both definitions are equivalent.
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One can go about developing an analogue of descriptive set theory in this context - defining $G_\delta$ gray sets, relative topological notions, stating (and proving) versions of the Baire category and Kuratowski–Ulam theorems for gray sets...
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For $A \subseteq X$, denote by $(A)_\partial$ the largest 1-Lipschitz map smaller than $A$, namely $(A)_\partial(y) = \inf A(x) + \partial(x, y)$.
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Theorem
Assume that, for any open $O$ in $X$ and $r > 0$, $(O)_{<r}$ is open in $X$ (which is true if $X$ is a Polish topometric group).

Corollary
Assume that $X$ satisfies the above assumption, and $A$ is 1-Lipschitz. Then $U(A)$ is also 1-Lipschitz.
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In particular, if a subset $A$ of $X$ is comeager in an open subset $O$, then $(A)_{<r}$ is comeager in $(O)_{<r}$ for all $r$.

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Assume that $X$ satisfies the above assumption, and $A$ is 1-Lipschitz. Then $U(A)$ is also 1-Lipschitz.
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Corollary
Assume that $X$ satisfies the above assumption, and $A$ is $1$-Lipschitz. Then $U(A)$ is also $1$-Lipschitz.
Theorem (Effros)

Let $G$ be a Polish group acting continuously on a Polish space $X$. Then $x \in X$ is generic iff $G \cdot x$ is dense and the map $g \mapsto g \cdot x$ is open (from $G$ to $G \cdot x$).
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When this holds, the orbit of $x$ must be $G_\delta$.

What is the topometric extension of this theorem?
**Theorem**

Let $G$ be a Polish group acting continuously on a Polish topometric space $(X, \tau, \partial)$. Assume that $(U)_r$ is open for any open $U$ and $r > 0$.

Then $x \in X$ is a metric generic iff $G \cdot x$ is dense and the map $g \mapsto g \cdot x$ is metrically open (from $G$ to $G \cdot x$), i.e.:

For any $\varepsilon > 0$, for any open $U \subseteq G$, $(U \cdot x)_\varepsilon$ is open in $G \cdot x$.

When this holds, $G \cdot x$ must be $G_\delta$. 
IV. Gray subgroups
From now on $G$ is a topological group whose topology is induced by a complete metric. We are used to multiplying subsets of $G$:

$$A \cdot B = \{g \in G : \exists a \in A \exists b \in B \ ab = g\} .$$

One can also set $A^{-1} = \{a^{-1} : a \in A\}$. 
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**Definition**

Given $A, B \subseteq G$, define

$$A \ast B(g) = \inf_{xy=g} A(x) + B(y).$$

Also, $A^{-1}(g) = A(g^{-1})$. 
We just extended the usual multiplication/inverse operations defined on subsets to gray subsets; basic tools of the descriptive set theory of Polish groups also adapt. An example:
We just extended the usual multiplication/inverse operations defined on subsets to gray subsets; basic tools of the descriptive set theory of Polish groups also adapt. An example:

**Proposition ("Pettis lemma")**

For any $A, B \subseteq G$, $U(A) \ast U(B) \subseteq A \ast B$.

In particular, if $A$ is Baire-measurable and nonmeager, then $(A \ast A^{-1})^\circ(1) = 0$. 
A subset $H$ of $G$ is a subgroup iff $H$ is nonempty and $H \cdot H^{-1} \subseteq H$.

For gray subsets, the corresponding conditions become $\inf(H) = 0$ and $H \ast H^{-1} \subseteq H$; this is the same as:
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For gray subsets, the corresponding conditions become $\inf(H) = 0$ and $H * H^{-1} \subseteq H$; this is the same as:

- $H(1) = 0$.
- $H = H^{-1}$.
- $\forall g, h \ H(gh) \leq H(g) + H(h)$. 
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For gray subsets, the corresponding conditions become $\inf(H) = 0$ and $H \star H^{-1} \subseteq H$; this is the same as:

- $H(1) = 0$.
- $H = H^{-1}$.
- $\forall g, h \ H(gh) \leq H(g) + H(h)$.

So subgroups correspond to seminorms or, equivalently, to left-invariant pseudometrics on $G$ (via the correspondence $H(\cdot) \leftrightarrow d(1, \cdot)$).
Thus the gray analogue of a (closed) subgroup is a (lower semi-continuous) left-invariant pseudometric. This is reinforced by the following model-theoretic observation.
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**Observation**

Let $\mathcal{M}$ be an $\aleph_0$-categorical Polish structure, and $H \subseteq Aut(\mathcal{M})$ be an open real-valued gray subgroup. Then there exists an imaginary $a \in M^{eq}$ such that $H(g) = d(a, g(a))$. 
What does the small index property become in the topometric context? Subgroups become left-invariant pseudometrics; index then should translate to density character of the corresponding quotient space.
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**Definition**
A Polish topometric group \( (G, \tau, \partial) \) has the **small density property** if any \( \partial \)-Baire measurable left-invariant pseudometric of density character \(< 2^{\aleph_0} \) is open.
What does the small index property become in the topometric context? Subgroups become left-invariant pseudometrics; index then should translate to density character of the corresponding quotient space.

**Definition**

A Polish topometric group \((G, \tau, \partial)\) has the small density property if any \(\partial\)-Baire measurable left-invariant pseudometric of density character \(< 2^{\aleph_0}\) is open.

The Baire-measurability above simply comes from the fact that we should impose some mild definability assumption - otherwise the uniform metric \(\partial\) plays no part in the definition.
Theorem (Ben Yaacov–M.)
Any Polish topometric group with ample generics has the small density property.
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Any Polish topometric group with ample generics has the small density property.
This is essentially a reformulation of the automatic continuity theorem we saw earlier in the talk - but the approach via gray sets makes the proof more transparent.
Thank you for your attention!