Lelek fan from a projective Fraïssé limit

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- Lelek fan L is a subfan of the Cantor fan with a dense set of endpoints in L

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- Lelek fan is unique: Any two subfans of the Cantor fan with dense set of endpoints are homeomorphic (Bula-Oversteegen 1990 and Charatonik 1989)

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- It is homeomorphic to: the complete Erdős space, the set of endpoints of the Julia set of the exponential map, the set of endpoints of the separable universal $\mathbb{R}$-tree. (Kawamura, Oversteegen, Tymchatyn)

Projective Fraïssé theory – setup

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2. A topological L-structure is a compact zero-dimensional second-countable space A equipped with closed relations $R_i^A, i \in I$ and continuous functions $f_j^A, j \in J$.
3. Epimorphisms are continuous surjections preserving the structure.

Projective Fraïssé family – definition

A family $\mathcal{F}$ of finite topological L-structure is a projective Fraïssé family if:

1. (F1) (joint projection property: JPP) for any $A, B \in \mathcal{F}$ there is $C \in \mathcal{F}$ and epimorphisms from C onto A and from C onto B;
2. (F2) (amalgamation property: AP) for $A, B_1, B_2 \in \mathcal{F}$ and any epimorphisms $\phi_1\colon B_1 \to A$ and $\phi_2\colon B_2 \to A$, there exist C, $\phi_3\colon C \to B_1$ and $\phi_4\colon C \to B_2$ such that $\phi_1 \circ \phi_3 = \phi_2 \circ \phi_4$.

Projective Fraïssé limit – definition

A topological L-structure $\mathbb{L}$ is a projective Fraïssé limit of $\mathcal{F}$ if the following three conditions hold:

1. (L1) (projective universality) for any $A \in \mathcal{F}$ there is an epimorphism from $\mathbb{L}$ onto A;
2. (L2) (projective ultrahomogeneity) for any $A \in \mathcal{F}$ and any epimorphisms $\phi_1 \colon \mathbb{L} \to A$ and $\phi_2 \colon \mathbb{L} \to A$ there exists an isomorphism $h \colon \mathbb{L} \to \mathbb{L}$ such that $\phi_2 = \phi_1 \circ h$;
3. (L3) for any finite discrete topological space X and any continuous function $f \colon \mathbb{L} \to X$ there is an $A \in \mathcal{F}$, an epimorphism $\phi \colon \mathbb{L} \to A$, and a function $f_0 \colon A \to X$ such that $f = f_0 \circ \phi$.

Projective Fraïssé limit – existence and uniqueness

Theorem (Irwin-Solecki)

Let $\mathcal{F}$ be a countable projective Fraïssé family of finite structures. Then:

1. *there exists a projective Fraïssé limit of $\mathcal{F}$;*
2. *any two projective Fraïssé limits are isomorphic.*

A simple example of a projective Fraïssé family

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Then the projective Fraïssé limit is the Cantor set.

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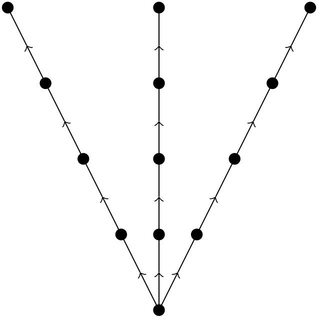

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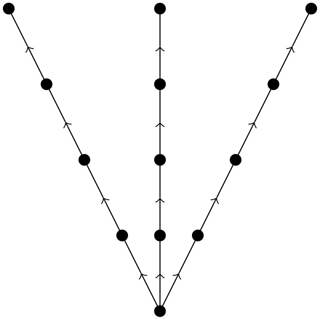

Theorem

$\mathcal{F}$ is a projective Fraïssé family.

Lelek fan from a projective Fraïssé limit, part 2

Lemma

Let $\mathbb{L}$ be the projective Fraïssé limit of $\mathcal{F}$. Then $R_S^{\mathbb{L}}$, where $R_S^{\mathbb{L}}(x, y)$ iff $R^{\mathbb{L}}(x, y)$ or $R^{\mathbb{L}}(y, x)$, is an equivalence relation such that each equivalence class has at most two elements.

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Theorem

$\mathbb{L}/R^{\mathbb{L}}$ is the Lelek fan.

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Theorem

1. *Each smooth fan is a continuous image of the Lelek fan.*
2. *Let X be a smooth fan. Let d be a metric on X. If f_1 and f_2 are continuous monotone surjections such that the top point goes to the top point, from the Lelek fan onto X, then for any $\epsilon > 0$ there exists a $h \in \mathrm{Aut}(\mathbb{L})$ of such that for all x, $d(f_1(x), f_2 \circ h(x)) < \epsilon$.*

Corollary

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The group $\mathrm{Aut}(\mathbb{L})$ *is dense in* $H(L)$.

Homeomorphism group of the Lelek fan–totally disconnected

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Proposition

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Fix a metric d on L. Denote the corresponding supremum metric on $H(L)$ by $d_{\sup}$.

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For every $\epsilon > 0$ and $h \in H(L)$, $h \neq Id$, there are ϵ-homeomorphisms $h_1, \ldots, h_n \in H(L)$ such that $h = h_1 \circ \ldots \circ h_n$.

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Theorem

For every $\epsilon > 0$ and $h \in H(L)$, $h \neq Id$, there are ϵ-homeomorphisms $h_1, \ldots, h_n \in H(L)$ such that $h = h_1 \circ \ldots \circ h_n$. Moreover, if $h \in \mathrm{Aut}(\mathbb{L})$, then we can choose required $h_1, \ldots, h_n$ in $\mathrm{Aut}(\mathbb{L})$.

Conjugacy classes of $H(L)$, part 1

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We use:

Theorem

The group of all automorphisms of $\mathbb{L}$, $\mathrm{Aut}(\mathbb{L})$, has a dense conjugacy class.

Conjugacy classes of $H(L)$, part 2

$$\mathcal{G} = \{(A, s^A) \colon A \in \mathcal{F} \text{ and there are } \phi \colon \mathbb{L} \to A \text{ and } f \in \mathrm{Aut}(\mathbb{L}) \text{ such that } \phi \colon (\mathbb{L}, \mathrm{graph}(f)) \to (A, s^A) \text{ is an epimorphism}\}.$$

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- $\mathcal{G}$ has the JPP if and only if for every $(A, s^A), (B, s^B) \in \mathcal{G}$ there is $(C, s^C) \in \mathcal{G}$ and epimorphisms from (C, s^C) onto (A, s^A) and onto (B, s^B).

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Proposition (K. , Kechris-Rosendal for (injective) Fraïssé)

The group $\mathrm{Aut}(\mathbb{L})$ has a dense cojugacy class if and only if $\mathcal{G}$ has the JPP.

Thank you!