Countable structures with simple automorphism groups

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Overview

Joint work with Zaniar Ghadernezhad and Katrin Tent

- Describe some general machinery for studying normal subgroups of automorphism groups of countable structures.
- Apply this to certain structures obtained using the Hrushovski amalgamation method.

1. Introduction

THEOREM (J. SCHREIER AND S. ULAM, 1933) Suppose X is countably infinite. If $g \in \mathrm{Sym}(X)$ moves infinitely many elements of X, then every element of $\mathrm{Sym}(X)$ is a product of conjugates of g. In particular, $\mathrm{Sym}(X)/\mathrm{FSym}(X)$ is a simple group.

THEOREM (A. ROSENBERG, 1958). Suppose V is a vector space of countably infinite dimension over a field K. If $FGL(V)$ denotes the elements of $GL(V)$ which have a fixed point space of finite codimension, then $GL(V)/(K^{\times}.FGL(V))$ is a simple group.

THEOREM (G. HIGMAN, 1954). The non-trivial, proper normal subgroups of $G = \mathrm{Aut}(\mathbb{Q}; \leq)$ are the left-bounded automorphisms, $L = \{g \in G : \exists a \; g|(-\infty, a) = id\}$, the right-bounded automorphisms $R = \{g \in G : \exists a \; g|(a, \infty) = id\}$ and $B = L \cap R$.

THEOREM (J. TRUSS, 1985). Let Γ be the countable random graph. Then $\mathrm{Aut}(\Gamma)$ is simple.

Warning

TEMPTING IDEA: Automorphism groups of 'nice' countable structures should not have any non-obvious normal subgroups.

This is false:

Example (M. Droste, C. Holland, D. Macpherson)

The automorphism group of a countable, homogeneous semilinear order has $2^{2^{\aleph_0}}$ normal subgroups.

A general result

Theorem (D. Lascar, 1992)

Suppose M is a countable saturated structure with a $\emptyset$-definable strongly minimal set D. Suppose that $M = \mathrm{acl}(D)$. Suppose $g \in G = \mathrm{Aut}(M/\mathrm{acl}(\emptyset))$ is unbounded, i.e. for every $n \in \mathbb{N}$ there is some $X \subseteq D$ with $\dim(gX/X) > n$. Then G is generated by the conjugates of g.

- Implies the results for $\mathrm{Sym}(X)$ and $GL(V)$.
- Proof uses Polish group arguments.
- Ideas used by T. Gardener (1995) to prove analogue of Rosenberg's result for classical groups over finite fields.
- Used by Z. Ghadernezhad and K. Tent (2012) to prove simplicity of automorphism groups of certain generalized polygons and so obtain new examples of simple groups with a BN-pair.

Recent general results

THEOREM (D. MACPHERSON AND K. TENT, 2011): Suppose M is a countable, transitive homogeneous relational structure whose age has free amalgamation. Suppose $\mathrm{Aut}(M) \neq \mathrm{Sym}(M)$. Then
(a) $\mathrm{Aut}(M)$ is simple;
(b) (Melleray) if $1 \neq g \in \mathrm{Aut}(M)$ then every element of G is a product of 32 conjugates of $g^{\pm 1}$.

NOTE: This implies Truss' result and unpublished results of M. Rubin (1988).

K. Tent and M. Ziegler (2012) generalized this to the case where M has a *stationary independence relation* $\perp$ and used this to prove:

THEOREM: Suppose U is the Urysohn rational metric space. If $g \in \mathrm{Aut}(U)$ is not bounded, then every automorphism of U is a product of 8 conjugates of g.

2. Stationary independence relations

NOTATION/ TERMINOLOGY:

- M is a countable first-order structure;
- $G = \mathrm{Aut}(M)$;
- cl is a G invariant, finitary closure operation on subsets of M;
- If $X \subseteq_{fin} M$ and a is fixed by G_X, then $a \in \mathrm{cl}(X)$ (where $G_X = \{g \in G : gx = x \ \forall x \in X\}$).
- $\mathcal{X} = \{\mathrm{cl}(A) : A \subseteq_{fin} M\}$;
- $\mathcal{F}$ consists of all maps $f : X \to Y$ with $X, Y \in \mathcal{X}$ which extend to automorphisms of M. Call these *partial automorphisms*.

EXAMPLE: Take cl to algebraic closure in M. So, for example, if M is the Fraïssé limit of a free amalgamation class, then $\mathrm{acl}(X) = X$ for all $X \subseteq M$.

In what follows, $\mathop{\smile\!\!\!\!\!\!|}$ is a relation between subsets A, B, C of M: written $A \mathop{\smile\!\!\!\!\!\!|}_B C$ and pronounced 'A is independent from C over B.'

Definition:

We say that $\mathop{\smile}\limits^{\mid}$ is a *stationary independence relation compatible with* cl if for $A, B, C, D \in \mathcal{X}$ and finite tuples a, b:

1. (Compatibility) We have $a \mathop{\smile}\limits^{\mid}_{b} C \Leftrightarrow a \mathop{\smile}\limits^{\mid}_{\mathrm{cl}(b)} C$ and
$$a \mathop{\smile}\limits^{\mid}_{B} C \Leftrightarrow e \mathop{\smile}\limits^{\mid}_{B} C \text{ for all } e \in \mathrm{cl}(a, B) \Leftrightarrow \mathrm{cl}(a, B) \mathop{\smile}\limits^{\mid}_{B} C.$$
2. (Invariance) If $g \in G$ and $A \mathop{\smile}\limits^{\mid}_{B} C$, then $gA \mathop{\smile}\limits^{\mid}_{gB} gC$.
3. (Monotonicity) If $A \mathop{\smile}\limits^{\mid}_{B} C \cup D$, then $A \mathop{\smile}\limits^{\mid}_{B} C$ and $A \mathop{\smile}\limits^{\mid}_{B \cup C} D$.
4. (Transitivity) If $A \mathop{\smile}\limits^{\mid}_{B} C$ and $A \mathop{\smile}\limits^{\mid}_{B \cup C} D$, then $A \mathop{\smile}\limits^{\mid}_{B} C \cup D$
5. (Symmetry) If $A \mathop{\smile}\limits^{\mid}_{B} C$, then $C \mathop{\smile}\limits^{\mid}_{B} A$.
6. (Existence) There is $g \in G_B$ with $g(A) \mathop{\smile}\limits^{\mid}_{B} C$.
7. (Stationarity) Suppose $A_1, A_2, B, C \in \mathcal{X}$ with $B \subseteq A_i$ and $A_i \mathop{\smile}\limits^{\mid}_{B} C$. Suppose $h : A_1 \to A_2$ is the identity on B and $h \in \mathcal{F}$. Then there is some $k \in \mathcal{F}$ which contains $h \cup \mathrm{id}_C$ (where id_C denotes the identity map on C).

Remarks and examples

1. For all $a \in M$ and finite X we have $a \mathop{\smile\!\!\!\!\!\mid}_X \mathrm{cl}(X)$. Moreover $a \mathop{\smile\!\!\!\!\!\mid}_X a$ iff $a \in \mathrm{cl}(X)$.
2. Tent and Ziegler consider this where $\mathrm{acl}(X) = X$ and $\mathrm{cl}(X) = X \ \forall X$. Write $A \mathop{\smile\!\!\!\!\!\mid}_B C$ to mean $A \cap C \subseteq B$. This satisfies (1-6), but not necessarily (7).
3. Suppose M is the Fraïssé limit of a free amalgamation class (of relational structures). Let $\mathrm{cl}(X) = X \ \forall X$. Define $A \mathop{\smile\!\!\!\!\!\mid}_B C$ to mean $A \cap C \subseteq B$ and $A \cup B$, $C \cup B$ are freely amalgamated over B. This is a stationary independence relation on M.
4. Suppose M is a countable-dimensional vector space over a countable field K. So $G = GL(M)$. Let cl be linear closure and take $A \mathop{\smile\!\!\!\!\!\mid}_B C$ to mean that $\mathrm{cl}(A \cup B) \cap \mathrm{cl}(C \cup B) = \mathrm{cl}(B)$. This gives a stationary independence relation.

Moving almost maximally

DEFINITION: Say that $g \in G$ moves almost maximally if for all $B \in \mathcal{X}$ and $a \in M$ there is a' in the G_B-orbit of a such that

$$a' \underset{B}{\perp} ga'.$$

EXAMPLE 1: Suppose $(M; \mathrm{cl}; \perp)$ is the vector space example. If $g \in G$ does not move almost maximally, then for some finite dimensional subspace B, for all $v \in M$ we have $gv \in \langle v, B \rangle$. Thus g acts as a scalar α on M/B. So $(\alpha^{-1}g - 1)v \in B$ for all v and it follows that g is a scalar multiple of a finitary transformation.

EXAMPLE 2: Suppose $(M; \mathrm{cl}; \perp)$ is the free amalgamation example. Suppose also that $G = \mathrm{Aut}(M)$ is transitive on M and $G \neq \mathrm{Sym}(M)$. If $1 \neq g \in G$, then g moves infinitely many points of each G_B-orbit (for each finite $B \subseteq M$) and using a back-and-forth argument, one shows that there is $h \in G$ such that $[g, h] = g^{-1}h^{-1}gh$ moves almost maximally.

Theorem A

Suppose M is a countable structure with a stationary independence relation compatible with a closure operation cl. Suppose that $G = \mathrm{Aut}(M)$ fixes every element of $\mathrm{cl}(\emptyset)$. If $g \in G$ moves almost maximally, then every element of G is a product of 16 conjugates of g.

REMARKS:

1. If $\mathrm{cl}(X) = X \ \forall X$, this is proved in the paper of Tent and Ziegler.
2. As observed by Tent and Ziegler, it implies the result of Macpherson and Tent for the free amalgamation example.
3. Following Lascar's paper, the proof uses the topology on G which has a base of open neighbourhoods of the form $\{g \in G : g(x) = f(x) \ \forall x \in X\}$ for $f : X \to Y$ a partial automorphism ($X, Y \in \mathcal{X}$). This is complete metrizable, but not necessarily separable. A trick from Lascar's paper allows one to work in separable closed subgroups and then the proof of Tent and Ziegler works.

3. Independence relations from dimension functions.

M a countable structure; $G = \mathrm{Aut}(M)$.

DEFINITION:

1. An integer-valued function d defined on finite subsets (or tuples) from M is a dimension function if for all finite $A, B \subseteq M$:
   (i) $d(gA) = d(A)$ for all $g \in G$;
   (ii) $0 \leq d(A) \leq d(A \cup B) \leq d(A) + d(B) - d(A \cap B)$.
2. Define $d(A/B) = d(A \cup B) - d(B)$ and for arbitrary C define $d(A/C) = \min(d(A/C') : C' \subseteq_{fin} C)$.
3. Define *d-closure* by: $\mathrm{cl}^d(C) = \{a \in M : d(a/C) = 0\}$.
4. Define d-independence by: $A \mathop{\smile\!\!\!\!\!\mid}^d_B C \Leftrightarrow d(A/B \cup C) = d(A/B)$.

LEMMA: cl^d is a finitary closure operation on M and $\mathop{\smile\!\!\!\!\!\mid}^d$ satisfies conditions (1-5) of being a stationary independence relation compatible with cl^d.

If conditions (6,7) (Existence, Stationarity) also hold, say that $\mathop{\smile\!\!\!\!\!\mid}^d$ is *stationary*.

Basic orbits

Suppose d is a dimension function on M. Let $\mathcal{X} = \{\mathrm{cl}^d(A) : A \subseteq_{fin} M\}$. If $b \in M$ and $C \in \mathcal{X}$ then $\mathrm{orb}(b/C) = \{gb : g \in G_C\}$ is the G_C-orbit of b over C.

DEFINITION: Suppose $b \in M$ and $A \in \mathcal{X}$. Say that b is *basic* over A if $b \notin A$ and whenever $A \subseteq C \in \mathcal{X}$ and $d(b/C) < d(b/A)$, then $b \in C$.

REMARKS:

1. If $d(b/A) = 1$, then b is basic over A.
2. If $b \notin A$ there is $A \subseteq C \in \mathcal{X}$ such that b is basic over C.
3. If $D \subseteq M$ is such that all $b \in D$ are basic over A, then d-closure over A gives a pregeometry on D.

DEFINITION: Say that $h \in G$ is *bounded* if there is some $C \in \mathcal{X}$ and $b \in M$ basic over C such that for all $b' \in \mathrm{orb}(b/C)$ we have

$$hb' \in \mathrm{cl}^d(C, b').$$

Theorem B

Suppose $\mathop{\smile\!\!\!\!\!\!\!|}^d$ is stationary and $A \in \mathcal{X}$ is such that there is a basic G_A-orbit D with $\mathrm{cl}^d(A \cup D) = M$. Suppose $g \in G = \mathrm{Aut}(M/\mathrm{cl}^d(\emptyset))$ is not bounded. Then every element of G is a product of 96 conjugates of $g^{\pm 1}$.

REMARKS:

1. Compare with Lascar's Theorem.
2. Proof:
   - There is a commutator $g' \in G_A$ of g which is not bounded.
   - g' moves almost maximally (over A)
   - So every element of G_A is a product of 32 conjugates of $g^{\pm 1}$ (Theorem A)
   - Take $h \in G$ with $hA \mathop{\smile\!\!\!\!\!\!\!|}^d A$. Then $G = G_A G_A^h G_A$.

4. Application to Hrushovski constructions.

- Integers $r \geq 2$ and $m, n \geq 1$.
- Language L has an r-ary relation symbol R.
- $\mathcal{C}$: class of r-uniform hypergraphs.
- If $B \in \mathcal{C}$, let $R[B] = \{\{b_1, \ldots, b_r\} : B \models R(b_1, \ldots, b_r)\}$.
- If B finite, the predimension of B is

$$\delta(B) = n|B| - m|R[B]|.$$

- If $B \subseteq_{fin} C \in \mathcal{C}$, write $B \leq C$ to mean $\delta(B) \leq \delta(B')$ for all $B \subseteq B' \subseteq_{fin} C$.
- In general, write $A \leq C$ to mean $A \cap X \leq X$ for all $X \subseteq_{fin} C$.
- If $A \leq B \leq C$ then $A \leq C$.
- $\mathcal{C}_0 = \{C \in \mathcal{C} : \emptyset \leq C\} = \{C \in \mathcal{C} : \delta(B) \geq 0 \ \forall B \subseteq_{fin} C\}$.

NOTE: If $X \subseteq_{fin} C \in \mathcal{C}_0$ there is some $X \subseteq Y \subseteq_{fin} C$ with $\delta(Y)$ as small as possible. Then $Y \leq C$.

The uncollapsed structure

LEMMA: If $A \leq B, C \in \mathcal{C}_0$, then the free amalgam F of B, C over A is in $\mathcal{C}_0$ and $B, C \leq F$.

Construction (Hrushovski, 'uncollapsed' case)

There is a countable $M_0 \in \mathcal{C}_0$, unique up to isomorphism, with the properties:

1. if $B \in \mathcal{C}_0^{fin}$ there is an embedding $f : B \to M_0$ with $f(B) \leq M_0$;
2. if $A_1, A_2 \leq M_0$ are finite and $h : A_1 \to A_2$ is an isomorphism, then h extends to an automorphism of M_0.

REMARK: M_0 is ω-stable, of infinite Morley rank.

Dimension in M_0

DEFINITION: If $X \subseteq_{fin} M_0$ define the dimension of X:

$$d(X) = \min(\delta(Y) : X \subseteq Y \subseteq_{fin} M_0).$$

OBSERVATIONS:

- This is a dimension function on M_0.
- In general, $\mathrm{cl}^d(X) = \{a \in M_0 : d(X \cup \{a\}) = d(X)\}$ is infinite and properly contains $\mathrm{acl}(X)$.

LEMMA: (1) Suppose $A_1, A_2 \in \mathcal{X}$ and $h : A_1 \to A_2$ is an isomorphism. Then h extends to an automorphism of M_0.
(2) Suppose $A_1, A_2, C \in \mathcal{X}$ and $B = A_i \cap C$. Suppose $A_i \cup C \leq M_0$ and A_i, C are freely amalgamated over B. If $h : A_1 \to A_2$ is an isomorphism which is the identity on B, then h extends to an automorphism of M_0 which is the identity on C.

COROLLARY: $\mathop{\smile\!\!\!\!\!\!|}\,^d$ is a stationary independence relation on M_0 compatible with cl^d.

Simplicity

RECALL: $r \geq 2$ arity of R; $\delta(B) = n|B| - m|R[B]|$.

Theorem (DE, Z. Ghadernezhad, K. Tent)

Suppose that if $r = 2$ then $n > m$, and if $r \geq 3$, then $n \geq m$. Let $G = \mathrm{Aut}(M_0/\mathrm{cl}^d(\emptyset))$. Then:

1. If $A \in \mathcal{X}$, $b \in M_0 \setminus A$ is basic over A, and $D = \mathrm{orb}(b/A)$, then $\mathrm{cl}^d(A, D) = M_0$.
2. If $k \in G$ is bounded, then $k = 1$.
3. If $1 \neq g \in G$, then every element of G is a product of 96 conjugates of $g^{\pm 1}$. In particular, G is a simple group.

Of course, (3) follows from (1,2) using the previous Corollary and Theorem B.

The ω-categorical construction

As in the previous section, this construction is due to Hrushovski. Suppose that if $r = 2$ then $n > m$ and if $r \geq 3$ then $n \geq m$.

Notation as before and $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ a continuous, increasing function with $f(x) \to \infty$ as $x \to \infty$. Let

$$\mathcal{C}_f = \{A \in \mathcal{C}_0 : \delta(X) \geq f(|X|)\ \forall X \subseteq_{fin} A\}.$$

If $A \subseteq B \in \mathcal{C}_f$ write $\mathrm{cl}^d_B(A) = \{b \in B : d(b/A) = 0\}$ and $A \leq_d B$ if $\mathrm{cl}^d_B(A) = A$.

REMARKS:

1. If $X \subseteq_{fin} A \in \mathcal{C}_f$ then $|\mathrm{cl}^d_A(X)| \leq f^{-1}(n|X|)$.
2. Say that f is *good* if the class $(\mathcal{C}^{fin}_f; \leq_d)$ has the free amlgamation property: if $A \leq_d B_1, B_2 \in \mathcal{C}^{fin}_f$, then the free amalgam E of B_1, B_2 over A is in $\mathcal{C}_f$.
3. If f is piecewise differentiable and the right derivative f' is increasing and satisfies $f'(x) \leq 1/x$ for $x \geq 1$, then f is good.

Construction (Hrushovski, ω-categorical case)

Suppose f is good. Then there is a countable $M_f \in \mathcal{C}_f$, unique up to isomorphism, such that:

1. if $B \in \mathcal{C}_f^{fin}$ there is an embedding $k : B \to M_f$ with $k(B) \leq_d M_f$;
2. if $A_1, A_2 \leq_d M_f$ are finite and $h : A_1 \to A_2$ is an isomorphism, then h extends to an automorphism of M_f.

REMARKS:

- M_f is ω-categorical.
- If $f(0) = 0$ and $f(1) = n$, then $\mathrm{cl}^d(\emptyset) = \emptyset$ and $\mathrm{Aut}(M_f)$ is transitive on M_f.

A stationary independence relation

The independence relation $\mathop{\smile\!\!\!\!\!\!\mid}^{d}$ is not stationary.

DEFINITION: If $A, B, C \subseteq_{fin} M_f$ write $A \perp_B C$ if:

- $A \mathop{\smile\!\!\!\!\!\!\mid}^{d}_{B} C$;
- $\mathrm{cl}^d(A \cup B \cup C) = \mathrm{cl}^d(A \cup B) \cup \mathrm{cl}^d(B \cup C)$.

LEMMA: The relation $\perp$ is a stationary independence relation on M_f, compatible with cl^d.

Theorem (DE, Z. Ghadernezhad, K. Tent)

Suppose f is good and $f(0) = 0$. Suppose M_f satsfies the following condition :

(*) if E is a basic orbit over $B \in \mathcal{X}$ then $M_f = \mathrm{cl}^d(B, E)$.

Let $1 \neq g \in G = \mathrm{Aut}(M_f)$ and $A \in \mathcal{X}$ be such that there is a basic G_A-orbit on M_f. Then:

1. There is $h_1 \in G$ such that $1 \neq g_1 = [g, h_1] \in G_A$.
2. g_1 moves almost maximally over A with respect to $\smile^d$.
3. There is $h_2 \in G_A$ such that $g_2 = [g_1, h_2]$ moves almost maximally with respect to $\perp$ over A.
4. Every element of G is a product of 192 conjugates of $g^{\pm 1}$.

Verifying (*)

Checking that (*) holds is surprisingly difficult and is only done for some special cases:

LEMMA: (1) Suppose $r = 2$ and $\delta(A) = 2|A| - |R[A]|$. Suppose $f(0) = 0, f(1) = 2, f(2) = 3$ and $f'(x) \leq 1/x$ for $x \geq 2$. Then f is good and M_f satisfies (*).
(2) Suppose $r \geq 3$ and $\delta(A) = |A| - |R[A]|$. Suppose $f(k) = k$ for $k < r$ and $f(r) = r - 1$. Suppose $f'(x) \leq 1/x$ for $x \geq r$. Then f is good and M_f satisfies (*).

So in these cases, $\mathrm{Aut}(M_f)$ is a simple group.

Problems

1. Is there a clean proof that (*) holds in most cases?
2. Look at the case where the predimension is given by $\alpha|A| - |R[A]|$, for irrational α. Do this for the 'uncollapsed' case and Hrushovski's ω-categorical version (appropriate α).
3. Small index property:
   - Is it true that if $H \leq \mathrm{Aut}(M_0)$ is of index $< 2^{\aleph_0}$ then there is $A \in \mathcal{X}$ with $H \geq G_A$?
   - If $A_1, A_2 \leq_d B \in \mathcal{C}_f^{fin}$ and $g : A_1 \to A_2$ is an isomorphism, is there $D \in \mathcal{C}_f^{fin}$ with $B \leq_d D$ and $h \in \mathrm{Aut}(D)$ with $h(a) = g(a) \, \forall a \in A_1$?