Convex duality in stochastic programming and mathematical finance

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Stochastic Optimization - Models and Algorithms,
Bonn, 27-29 May 2013

Convex duality in stochastic programming

- We describe a duality framework that unifies various duality arguments in stochastic programming and mathematical finance in finite discrete time.
- This is done by combining the conjugate duality framework of [Rockafellar 1974] with certain measure theoretic techniques from mathematical finance.
- The extension allows for
  - closing the duality gap in many models where traditional topological arguments fail.
  - extending various results of mathematical finance to nonlinear market models with portfolio constraints and illiquidity effects.

Convex duality in stochastic programming

- Rockafellar and Wets, *Nonanticipativity and L^1-martingales in stochastic optimization problems*, Math. Progr., 1976
- Dalang, Morton and Willinger, *Equivalent martingale measures and no-arbitrage in stochastic securities market models*, Stoch. and Stoch. Rep., 1990.
- Schachermayer, *A Hilbert space proof of the fundamental theorem of asset pricing in finite discrete time*, Insurance Math. Econ., 1992
- Pennanen and Penner, *Hedging of claims with physical delivery under convex transaction costs*, SIAM J. Fin. Math., 2010.
- Pennanen, *Convex duality in stochastic programming and mathematical finance*, Math. Oper. Res., 2011.
- Pennanen and Perkkiö, *Stochastic programs without duality gaps*, Math. Progr., 2012.

The problem

- Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, P)$ be a filtered probability space and

$$\mathcal{N} = \{(x_t)_{t=0}^T \,|\, x_t \in L^0(\Omega, \mathcal{F}_t, P; \mathbb{R}^{n_t})\}$$

  for some integers n_t.
- Let $f : \mathbb{R}^{n_0} \times \cdots \times \mathbb{R}^{n_T} \times \mathbb{R}^m \times \Omega \to \overline{\mathbb{R}}$ be a convex normal integrand.
- We will study the parametric optimization problem

$$\underset{x \in \mathcal{N}}{\text{minimize}} \quad Ef(x,u) := \int f(x(\omega), u(\omega), \omega) dP(\omega),$$

  where $u \in L^p(\Omega, \mathcal{F}, P; \mathbb{R}^m)$ and $Ef(x,u) := +\infty$ unless the positive part of the integrand is integrable.
- We denote the optimal value by $\varphi(u) = \inf_{x \in \mathcal{N}} Ef(x,u)$.

Example 1 (Optimal stopping) *If $n_t = m = 1$ and*

$$f(x,u,\omega) = \begin{cases} -\sum_{t=0}^T x_t Z_t(\omega) & \text{if } x \ge 0 \text{ and } \sum_{t=0}^T x_t + u \le 0, \\ +\infty & \text{otherwise,} \end{cases}$$

for an adapted real-valued process Z, the problem becomes

$$\operatorname*{minimize}_{x \in \mathcal{N}_+} \quad E\sum_{t=0}^T x_t Z_t \quad \text{subject to} \quad \sum_{t=0}^T x_t + u \le 0 \ P\text{-a.s.}$$

When $u = -1$, this is a convex relaxation of the optimal stopping problem. The relaxation does not affect the optimal value $\varphi(u)$.

The problem

Example 2 (Shadow price of information) *If $m = n$ and*

$$f(x, u, \omega) = h(x + u, \omega),$$

where h is a convex normal integrand, then $\varphi(u)$ is the optimal value of the nonadapted perturbation

$$\underset{x \in \mathcal{N}}{\text{minimize}} \quad Eh(x + u).$$

of the stochastic optimization problem

$$\underset{x \in \mathcal{N}}{\text{minimize}} \quad Eh(x).$$

The problem

Example 3 (Optimal investment) *Let $n_t = d$, $m = 1$ and*

$$f(x,u,\omega) = \begin{cases} v\left(u - \sum_{t=0}^{T-1} x_t \cdot \Delta S_{t+1}(\omega)\right) & \text{if } x_t \in D_t(\omega), \\ +\infty & \text{otherwise} \end{cases}$$

where $v : \mathbb{R} \to \overline{\mathbb{R}}$ is convex, S is an adapted price process and $\omega \mapsto D_t(\omega)$ are convex and measurable. The problem becomes

$$\underset{x \in \mathcal{N}}{\text{minimize}} \quad Ev\left(u - \sum_{t=0}^{T-1} x_t \cdot \Delta S_{t+1}\right) \quad \text{s.t.} \quad x_t \in D_t \; a.s.$$

If $v = \delta_{\mathbb{R}_-}$, we get $\varphi = \delta_{\mathcal{C}}$, where

$$\mathcal{C} = \{u \in L^0 \,|\, \exists x \in \mathcal{N} : \sum_{t=0}^{T-1} x_t \cdot \Delta S_{t+1} \geq u\}.$$

The problem

More examples:

- Transaction costs,
- Optimal consumption,
- Pricing of financial contracts,
- von Neumann-Gale model [Evstigneev and Schenk-Hoppé, 2006]
- ...

Duality

- Our aim is to derive dual expressions for the value function

$$\varphi(u) = \inf_{x \in \mathcal{N}} Ef(x, u)$$

on $L^p := L^p(\Omega, \mathcal{F}, P; \mathbb{R}^m)$.

- The bilinear form

$$\langle u, y \rangle = Eu(\omega) \cdot y(\omega)$$

puts L^p and L^q in separating duality.

- The conjugate of φ is defined by

$$\varphi^*(y) = \sup_{u \in L^p} \{\langle u, y \rangle - \varphi(u)\}.$$

- If φ is proper and lower semicontinuous, then $\varphi^{**} = \varphi$.

Duality

- The associated Lagrangian is the convex-concave function $L : \mathcal{N} \times L^q \to \overline{\mathbb{R}}$ defined by

$$L(x, y) = \inf_{u \in L^p} \{Ef(x, u) - \langle u, y \rangle\}.$$

- The conjugate of $\varphi(u)$ can be expressed as

$$\begin{aligned} \varphi^*(y) &= \sup\{\langle u, y \rangle - \inf_{x \in \mathcal{N}} Ef(x, u)\} \\ &= -\inf_{x \in \mathcal{N}} L(x, y). \end{aligned}$$

- We deviate from the conjugate duality framework of [Rockafellar, 1974] only in that $\mathcal{N}$ does not have a locally convex topology.

Theorem 4 *Assume that $Ef \not\equiv +\infty$. Then*

$$L(x,y) = El(x,y) \quad \forall x \in \mathcal{N}: \; Ef(x,\cdot) \not\equiv +\infty,$$

where $l(x,y,\omega) = \inf_{u\in\mathbb{R}^m}\{f(x,u,\omega) - u\cdot y\}$. *If*

$$l(x,y,\omega) = \sum_{t=0}^{T} l_t(x_t,y,\omega)$$

for some $\mathcal{B}(\mathbb{R}^{n_t}) \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{F}$*-measurable functions* l_t *on* $\mathbb{R}^{n_t} \times \mathbb{R}^m \times \Omega$ *then*

$$\varphi^*(y) = -\inf_{x\in\mathcal{N}^\infty} El(x,y) \qquad \textit{where } \mathcal{N}^\infty := \mathcal{N} \cap L^\infty$$

as long as the infimum above is less than $+\infty$.

Examples

Example 5 (Optimal stopping) *When*

$$f(x,u,\omega)=\begin{cases}-\sum_{t=0}^T x_tZ_t(\omega) & \text{if } x\ge 0 \text{ and } \sum_{t=0}^T x_t+u\le 0,\\ +\infty & \text{otherwise,}\end{cases}$$

we get

$$l(x,y,\omega)=\begin{cases}+\infty & \text{unless } x\ge 0,\\ \sum_{t=0}^T x_t(y-Z_t(\omega)) & \text{if } x\ge 0 \text{ and } y\ge 0,\\ -\infty & \text{otherwise.}\end{cases}$$

By Theorem 1,

$$\begin{aligned}-\varphi^*(y)&=\inf_{\mathcal{N}_+^\infty} E\sum_{t=0}^T x_t(y-Z_t)=\inf_{\mathcal{N}_+^\infty} E\sum_{t=0}^T E_t[x_t(y-Z_t)]\\ &=\inf_{\mathcal{N}_+^\infty} E\sum_{t=0}^T x_t(E_ty-Z_t)=E\sum_{t=0}^T \inf_{x_t\in\mathbb{R}_+} x_t(E_ty-Z_t).\end{aligned}$$

Examples

Example 6 (Optimal stopping, cont.) *Thus*

$$\varphi^*(y) = \begin{cases} 0 & \text{if } y \geq 0 \text{ and } E_t y \geq Z_t,\ t = 0, \dots, T \\ +\infty & \text{otherwise.} \end{cases}$$

and

$$\varphi^{**}(u) = \sup_{y \in L^1_+} \{Euy \,|\, E_t y \geq Z_t\} \quad \forall u \in L^\infty.$$

In particular,

$$\begin{aligned} \varphi^{**}(-1) &= -\inf\{Ey \,|\, E_t y \geq Z_t\} \\ &= -\inf_{y \in \mathcal{M}} \{y_0 \,|\, y_t \geq Z_t\}, \end{aligned}$$

where $\mathcal{M}$ is the space of martingales. The optimal y is the martingale part of the Snell envelope of Z_t.

Examples

Example 7 (Shadow price of information) *When* $f(x,u,\omega) = h(x+u,\omega)$, *we get*

$$l(x,y,\omega) = x\cdot y - h^*(y,\omega),$$

and

$$\varphi^*(y) = \begin{cases} Eh^*(y) & \text{if } E_t y_t = 0\ \forall t,\\ \infty & \text{otherwise.}\end{cases}$$

Thus,

$$\varphi^{**}(u) = \sup_{y\in\mathcal{N}^\perp} E[u\cdot y - h^*(y)]$$

In particular, $\varphi^{**}(0) = \sup_{y\in\mathcal{N}^\perp} E \inf_{x\in\mathbb{R}^n}\{h(x,\omega) - x\cdot y(\omega)\}$*; see [Rockafellar and Wets, 1976], [Back and Pliska, 1987] and [Davis, 1992].*

Example 8 (Optimal investment) *When*

$$f(x,u,\omega) = \begin{cases} v\left(u - \sum_{t=0}^{T-1} x_t \cdot \Delta S_{t+1}(\omega)\right) & \text{if } x_t \in D_t(\omega), \\ +\infty & \text{otherwise} \end{cases}$$

we get

$$l(x,y,\omega) = \sum_{t=0}^{T-1} \left\{\delta_{D_t(\omega)}(x_t) - x_t \cdot [y\Delta S_{t+1}(\omega)]\right\} - v^*(y).$$

and

$$\varphi^*(y) = E\left\{v^*(y) + \sigma_{D_t}(E_t[y\Delta S_{t+1}])\right\}$$

If $D_t(\omega) = \mathbb{R}^d$, we have $\varphi^*(y) = Ev^*(y) + \delta_{\mathcal{C}^*}(y)$, where $\mathcal{C}^*$ is the set of positive multiples of martingale densities. In particular, $\varphi^{**}(0) = -\inf\{Ev^*(y) \mid y \in \mathcal{C}^*\}$; see [Kramkov and Schachermayer, 1999].

Closedness criteria

- The above expressions for φ^{**} provide dual representations of the optimal value φ provided φ is proper and lower semicontinuous (lsc), i.e.

$$\liminf_{\nu\to\infty} \varphi(u^\nu) \geq \varphi(u)$$

  whenever $u^\nu \to u$ in L^1.
- The traditional "direct method" assumes that Ef is jointly lsc and $Ef(\cdot, u)$ is inf-compact uniformly locally in u.
- In financial models, the topological inf-compactness condition often fails but there is a more general measure theoretic counterpart.

Closedness criteria

Theorem 9 (Komlós') *Let $(x^\nu)_{\nu=1}^\infty$ be a sequence in $L^0(\Omega, \mathcal{F}, P; \mathbb{R}^n)$ which is almost surely bounded, i.e.*

$$\sup_\nu |x^\nu(\omega)| < \infty \quad P\text{-a.s.}$$

Then there is a sequence of convex combinations $\bar{x}^\nu \in \operatorname{co}\{x^\mu \,|\, \mu \geq \nu\}$ that converges almost surely to an $\mathbb{R}^n$-valued function.

Theorem 10 *Let $C(\omega) \subset \mathbb{R}^n$ be closed convex and $\mathcal{F}$-measurable. Every sequence in $\{x \in \mathcal{N} \,|\, x \in C \ a.s.\}$ is almost surely bounded if $\{x \in \mathcal{N} \,|\, x \in C^\infty \ a.s.\} = \{0\}$. Here,*

$$C^\infty(\omega) := \bigcap_{\alpha > 0} \alpha C(\omega).$$

Closedness criteria

Theorem 11 *Assume that f is bounded from below and that*

$$\{x \in \mathcal{N} \,|\, f^\infty(x(\omega), 0, \omega) \le 0 \text{ a.s.}\}$$

is a linear space. Then

$$\varphi(u) = \inf_{x \in \mathcal{N}} Ef(x, u)$$

is lsc on L^p and the inf is attained for every $u \in L^p$. Here,

$$f^\infty(x, u, \omega) := \sup_{\alpha > 0} \frac{f(\alpha x, \alpha u, \omega)}{\alpha}.$$

Closedness criteria

Example 12 (Optimal stopping) *When*

$$f(x,u,\omega) = \begin{cases} -\sum_{t=0}^{T} x_t Z_t(\omega) & \text{if } x \geq 0 \text{ and } \sum_{t=0}^{T} x_t \leq u, \\ +\infty & \text{otherwise,} \end{cases}$$

we have $f^\infty = f$ *and*

$$\{x \in \mathcal{N} \,|\, f^\infty(x,0) \leq 0 \text{ a.s.}\} = \{0\},$$

so the linearity condition is always satisfied.

Closedness criteria

Example 13 (Shadow price of information) *When*

$$f(x,u,\omega) = h(x+u,\omega),$$

the linearity condition means that

$$\{x \in \mathcal{N} \mid h^\infty(x) \le 0 \text{ -a.s.}\}$$

is linear.

Closedness criteria

Example 14 (Optimal investment) *When*

$$f(x,u,\omega) = \begin{cases} v\left(u - \sum_{t=0}^{T-1} x_t \cdot \Delta S_{t+1}(\omega)\right) & \textit{if } x_t \in D_t(\omega), \\ +\infty & \textit{otherwise} \end{cases}$$

we get

$$f^\infty(x,u,\omega) = \begin{cases} v^\infty\left(u - \sum_{t=0}^{T-1} x_t \cdot \Delta S_{t+1}(\omega)\right) & \textit{if } x_t \in D_t^\infty(\omega) \\ +\infty & \textit{otherwise.} \end{cases}$$

If v is nonconstant and $D_t(\omega) = \mathbb{R}^J$, the linearity condition becomes the no-arbitrage condition

$$x \in \mathcal{N} : \sum x_t \cdot \Delta S_{t+1} \geq 0 \implies \sum x_t \cdot \Delta S_{t+1} = 0.$$

With transaction costs, we get the robust no-arbitrage condition introduced by [Schachermayer, 2004].

Open problems

- Discrete-time models
  - The lower bound on the normal integrand f excludes e.g. utility functions that are unbounded from above.
  - Is there a general form of f that allows for scenariowise minimization in

$$-\varphi^*(y) = \inf_{x \in \mathcal{N}} El(x, y)$$

- Continuous-time models:
  - Scenariowise description of the wealth process as a function of the trading strategy (Ito-Föllmer integral?)
  - The closedness criterion in discrete time is based on an induction argument on the number of time steps.
  - Can the no free lunch with vanishing risk condition be extended to general normal integrands.