Countably categorical structures and the computational complexity of constraint satisfaction problems

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Constraint Satisfaction Problems

Let Γ be a structure with a finite relational signature τ.
Γ also called the template.

Definition 1 (CSP).

CSP(Γ) is the computational problem to decide whether a given finite τ-structure homomorphically maps to Γ.

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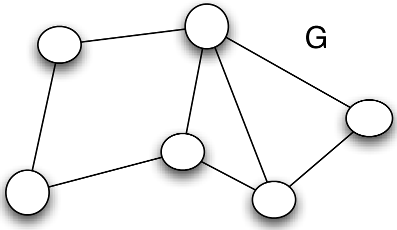
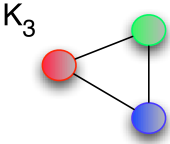
Example: 3-colorability is CSP(K_3)





Examples of CSPs

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Complexity: NP-complete.

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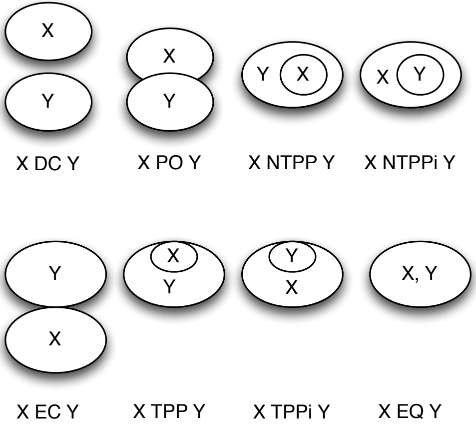
Abstract description of ‘regions’ by binary relations



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Moreover:

$$\begin{aligned}
\mathrm{PP}(x,y) &\Leftrightarrow \mathrm{NTPP}(x,y) \vee \mathrm{TPP}(x,y)\\
\mathrm{P}(x,y) &\Leftrightarrow \mathrm{PP}(x,y) \vee \mathrm{EQ}(x,y)\\
\mathrm{DC}(x,y) \wedge \mathrm{P}(z,y) &\Rightarrow \mathrm{DC}(x,z)\\
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\mathrm{Pi}(x,y) \wedge \mathrm{P}(y,z) &\Rightarrow \neg\mathrm{DC}(x,z)
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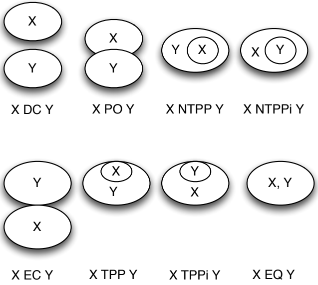

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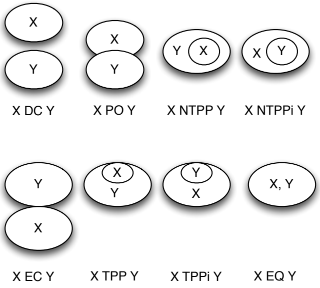

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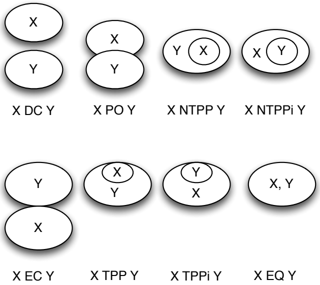

- **Fact:** the class of all finite RCC8 models forms an amalgamation class
- Let Γ be the Fraïssé-limit of this class
- CSP(Γ) studied intensively in Artificial Intelligence
  (for 'qualitative spatial reasoning'); can be solved in polynomial time

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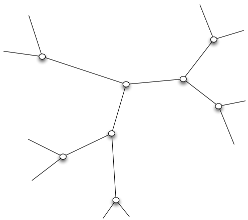

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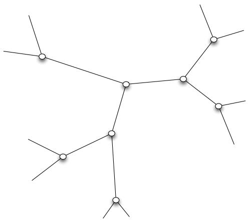

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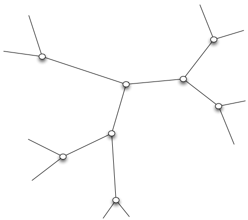

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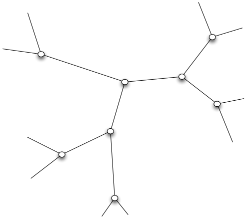

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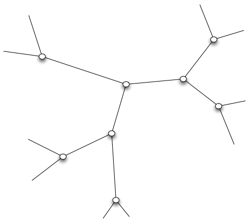

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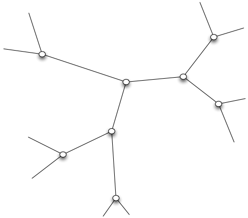

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Complexity Classification

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Given a fixed structure Γ,
what is the computational complexity of CSP(Γ)?

The Feder-Vardi Dichotomy Conjecture

What can be said about CSP(Γ) when Γ has a finite domain?

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Strongest evidence comes from the so-called universal algebraic approach.

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- Fraïssé-limits of classes of structures with finite relational signature are ω-categorical

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Remainder of the talk:

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- Open problems

1st Step: Primitive Positive Definability

A first-order formula is called primitive positive (pp) if it is of the form

$$\exists x_1, \ldots, x_n \, (\psi_1 \wedge \cdots \wedge \psi_m) \, ,$$

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How do we recognize whether a relation R is pp-definable in Γ?

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A function $f \colon D^k \to D$ preserves $R \subseteq D^m$ if
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Theorem (MB+Nešetřil'03).

Let Γ be ω-categorical. Then a relation R has a pp-definition in Γ if and only if R is preserved by all polymorphisms of Γ.

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The set of all polymorphisms of Γ, denoted $\mathrm{Pol}(\Gamma)$, is a function clone:

- closed under compositions
- contains the projections

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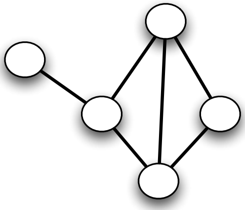

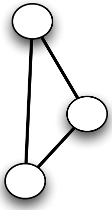

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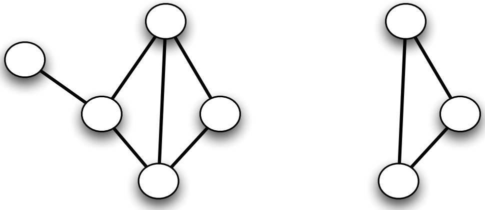

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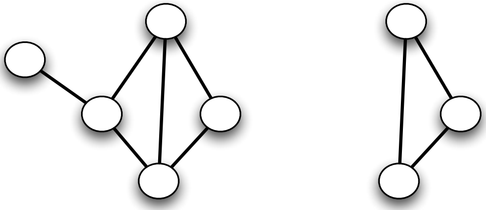

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Observation: When Γ is ω-categorical then the theory of Γ is model-complete if and only if all self-embeddings of Γ preserves all first-order formulas.

3rd Step continued: Model-complete Cores

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Nice properties of the model-complete core Δ:

- Aut(Δ) is dense in End(Δ)
  (the endomorphisms of Δ equipped with the topology of pointwise convergence)

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Every finite or countable ω-categorical structure is homomorphically equivalent to a model-complete core Δ. The structure Δ is unique up to isomorphism, and ω-categorical.

Nice properties of the model-complete core Δ:

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3rd Step continued: Model-complete Cores

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Example: The model-complete core of the countable random graph is a countably infinite clique.

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Then $\xi \colon \mathcal{C} \to \mathcal{D}$ is called a clone homomorphism if ξ preserves arities, preserves the projections, and

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Proposition.

Let Γ be ω-categorical such that $\mathrm{Pol}(\Gamma)$ homomorphically and continuously maps to the clone of projections. Then $\mathrm{CSP}(\Gamma)$ is NP-hard.

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Tractability is already known when there is a polymorphism that is

- majority, that is, satisfies $\forall x, y.\ f(x, x, y) = f(x, y, x) = f(y, x, x) = x$.
- Maltsev, that is, satisfies $\forall x, y.\ f(x, y, y) = f(y, y, x) = x$.
- semi-lattice, that is, is binary commutative, associative, idempotent.
- ...

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Observation: If Γ is homogeneous in a finite relational language, then there is only a finite number of distinct behaviors with respect to Γ.

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This method can also be applied to study reducts up to first-order interdefinability (see talk of Pongrácz!)

Proof Method Continued: Ramsey Classes

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Write $\binom{G}{H}$ for the set of all induced substructures of G that are isomorphic to H.

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Example: The class of all finite linearly ordered graphs. (Nešetřil-Rödl)

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Lemma (MB,Pinsker,Tsankov'11).

Let Δ be ω-categorical ordered Ramsey. Let $f \colon \Delta^k \to \Delta$, and $c_1, \dots, c_n \in \Delta$. Then there is a function in

$$S := \overline{\{\beta f(\alpha_1, \dots, \alpha_k) \mid \beta, \alpha_1, \dots, \alpha_k \in \mathrm{Aut}(\Delta, c_1, \dots, c_n)\}}$$

which is canonical with respect to $(\Delta, c_1, \dots, c_n)$.

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Cf. conjectures in other talks at the HIM

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Conclusion: since there are only countably many algorithms, there exists a Γ such that $\mathrm{CSP}(\Gamma)$ is undecidable.

There are even homogeneous digraphs Γ such that $\mathrm{CSP}(\Gamma)$ is in coNP, but neither coNP-complete, nor in P (unless P=coNP; MB.+Grohe'08)

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Conjecture (MB+Pinsker'11).

Reducts of homogeneous finitely bounded structures have a complexity dichotomy for the CSP: the CSP is NP-complete or in P.

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3. Is there an ω-categorical structure whose CSP is in NP, but neither NP-complete nor in P?