Finding a Stable Allocation in Polymatroid Intersection

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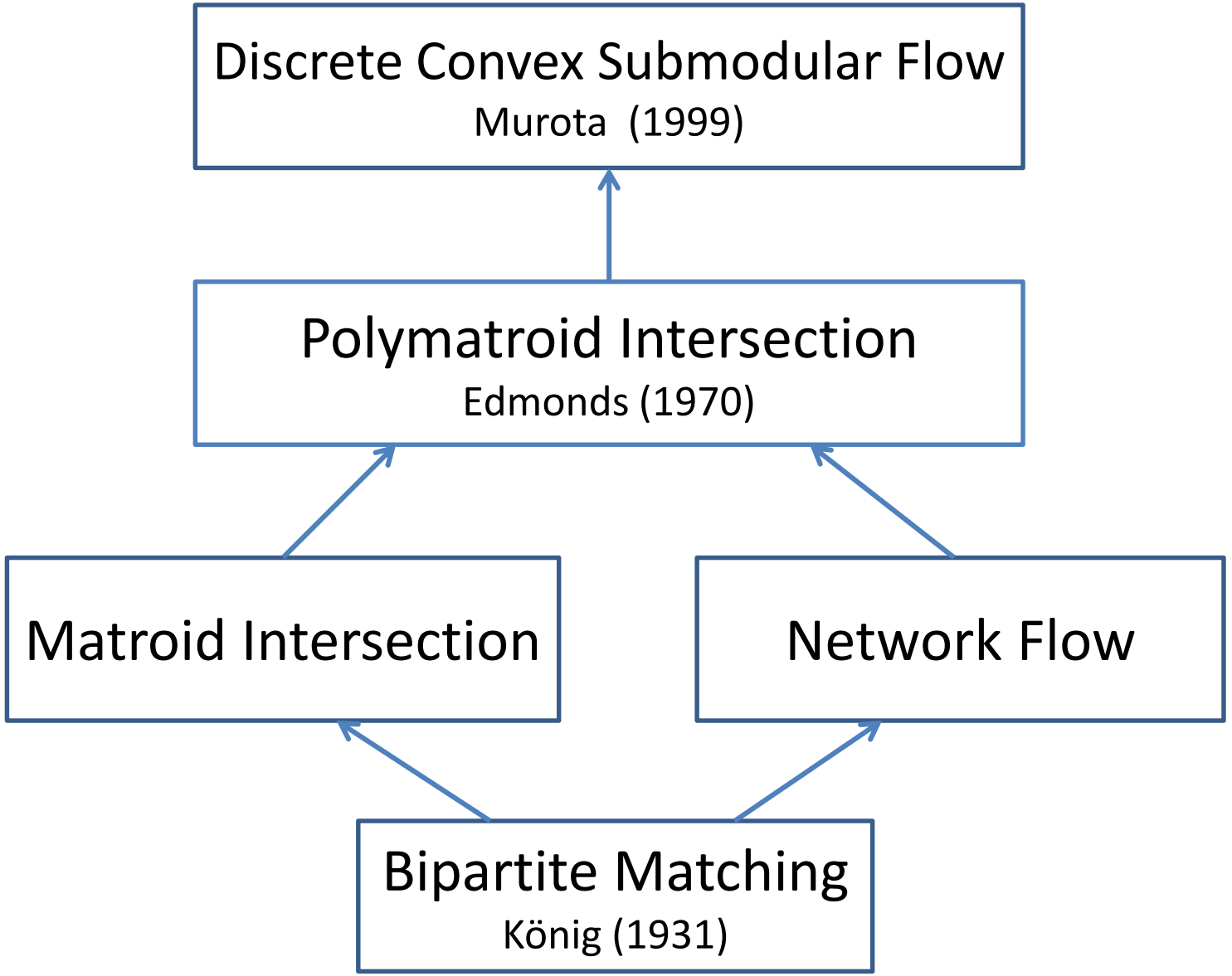
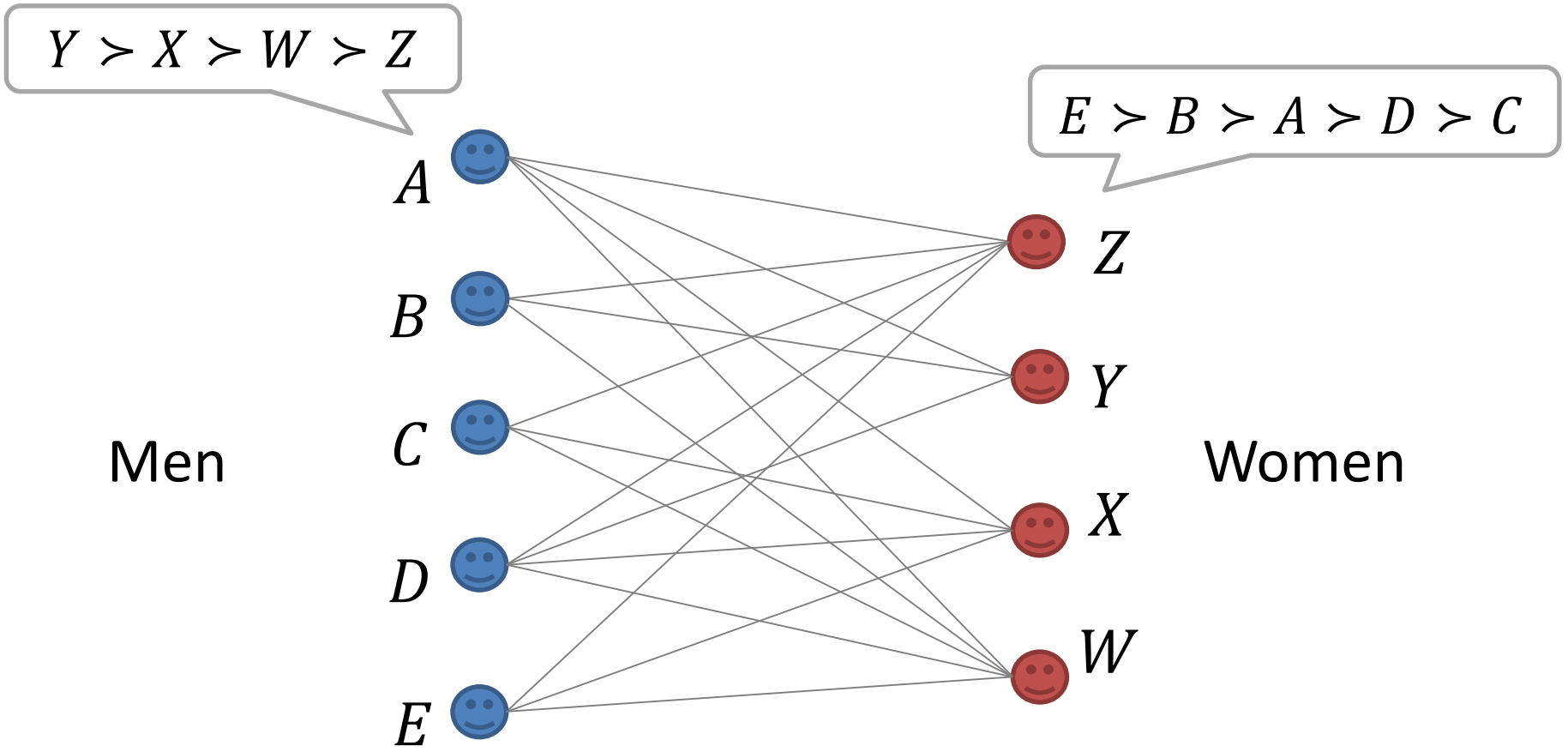
Generalizations of Bipartite Matching



Stable Matching Model

[Gale, Shapley 62]

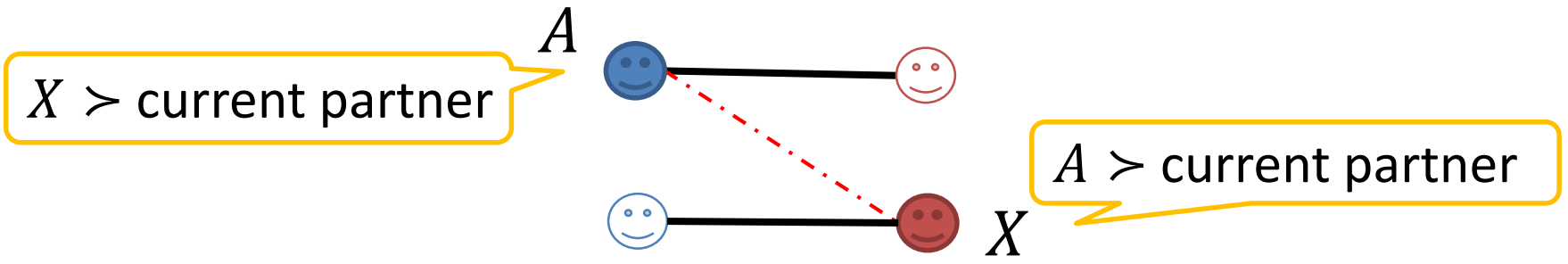


Goal: find a stable matching

Stability

A matching is **stable** $\overset{\text{def}}{\Leftrightarrow}$ there is no blocking pair

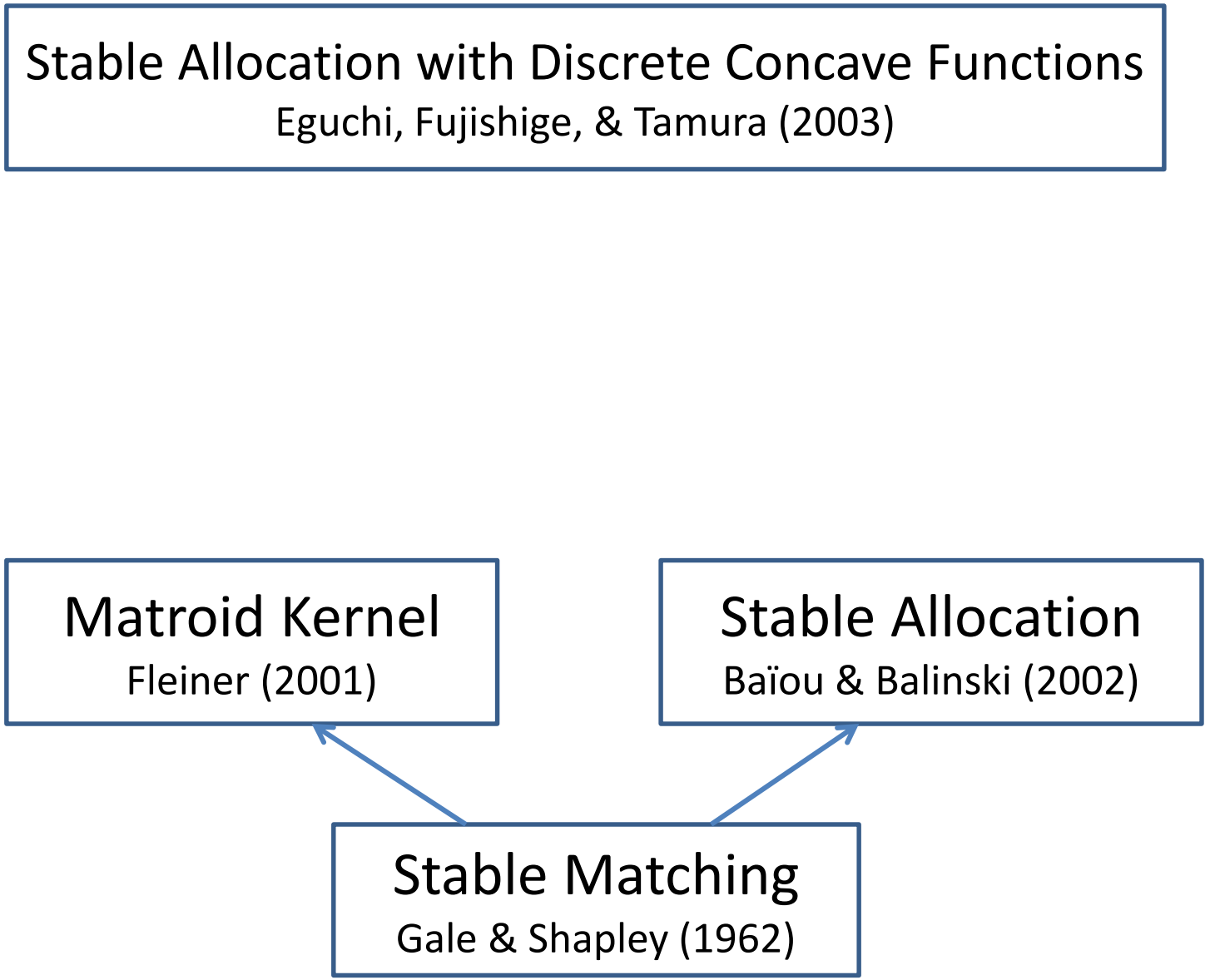
A pair currently unmatched but both want to be matched



(A, X) is a blocking pair

A stable matching can be found in $O(|E|)$ by the Gale-Shapley algorithm

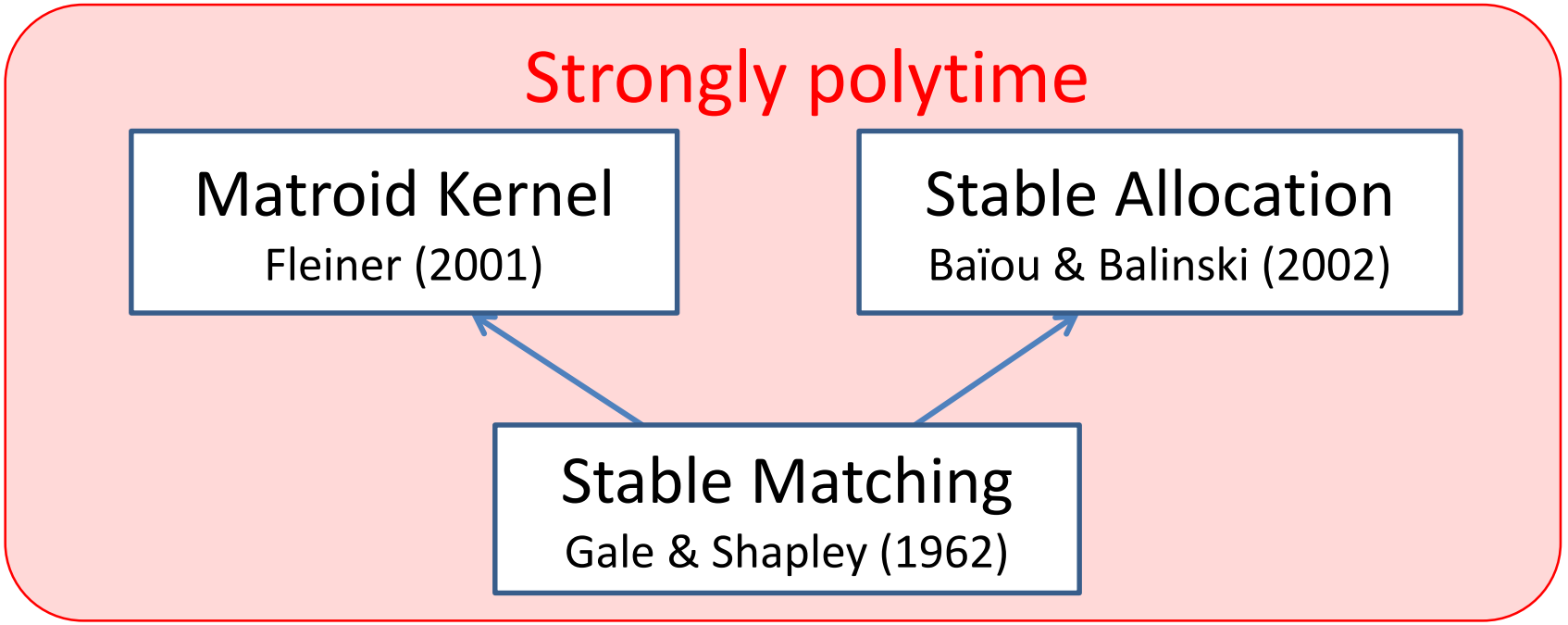
Generalizations of Stable Matching



Generalizations of Stable Matching

Stable Allocation with Discrete Concave Functions
Eguchi, Fujishige, & Tamura (2003)

Only pseudo-polynomial time algo. is known



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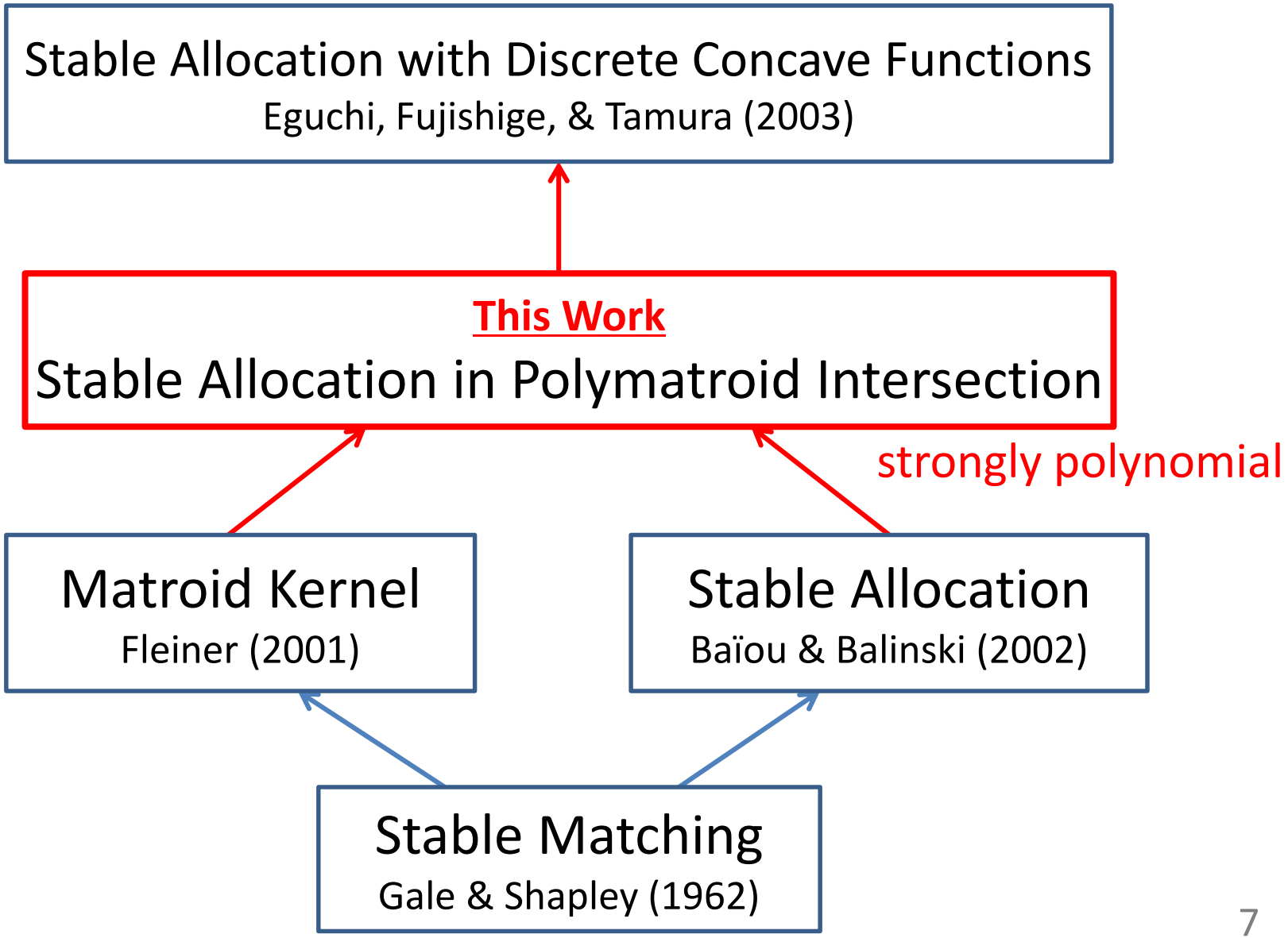

Algorithms for Related Models

- Matroid Kernel [Fleiner, 2001]

  Extended Gale-Shapley algo. strongly polynomial

- Stable Allocation [Baïou & Balinski, 2002]

  Extended Gale-Shapley algo. pseudo-polynomial

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  Augmenting Path Algorithm $O(|V| \cdot |E|)$

  $O(|E|\log|V|)$ [Dean & Munshi, 2010]

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- Discrete Concave Function Model [EFT, 2003]

  Extended Gale-Shapley algo. pseudo-polynomial

Polymatroids

E: Finite Set $\quad f: 2^E \to \mathbf{R}$

(E, f): Polymatroid

- $f(\emptyset) = 0$
- $A \subseteq B \Longrightarrow f(A) \leq f(B)$
- $\forall A, B \subseteq E$: $\quad f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$

Examples

Matroid Rank Functions

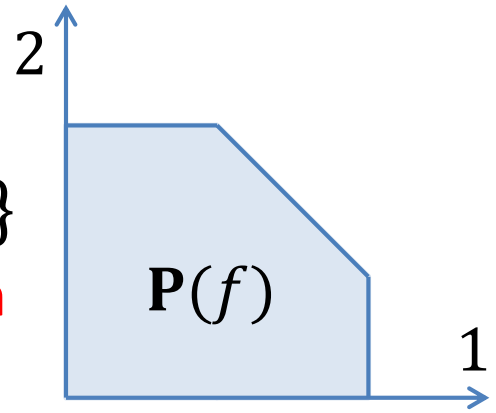
Coverage Functions

Entropy Functions

Polymatroids

ex. $E = \{1, 2\}$

(E, f): Polymatroid



$$\mathbf{P}(f) \coloneqq \{x \in \mathbf{R}_+^E \mid \forall A \subseteq E\colon\ x(A) \le f(A)\ \}$$

Polymatroid polyhedron

$x \in \mathbf{P}(f)$

$$\mathrm{sat}_f(x) \coloneqq \{u \in E \mid \forall \alpha > 0\colon\ x + \alpha\chi_u \notin \mathbf{P}(f)\}$$

Saturated elements

$x \in \mathbf{P}(f),\ u \in \mathrm{sat}_f(x)$

$$\mathrm{dep}_f(x, u) \coloneqq \{v \in E \mid \exists \alpha > 0\colon\ x + \alpha(\chi_u - \chi_v) \in \mathbf{P}(f)\}$$

Exchangeable elements for u

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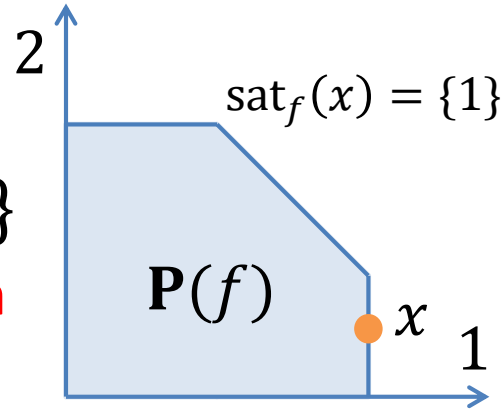

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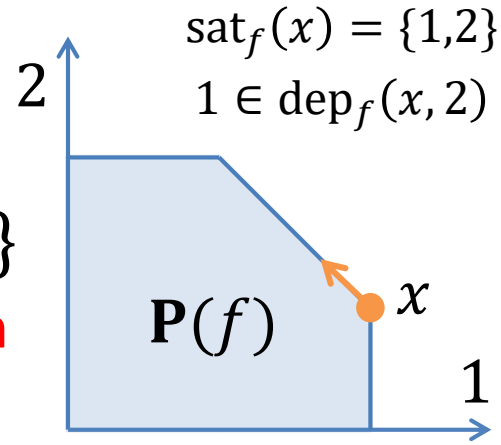

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Exchangeable elements for u

Ordered Polymatroids

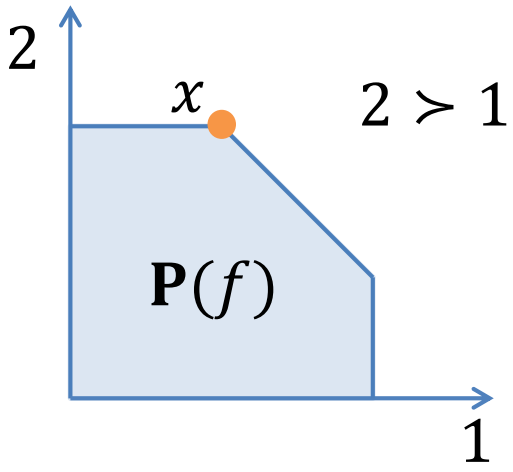
(E, f): Polymatroid

$(E, \succ)$: Total Order

$\Rightarrow$ $(E, f, \succ)$: Ordered Polymatroid

$x \in \mathbf{P}(f)$: Optimal $\Leftrightarrow$ $\forall u \in E,$ $u \in \mathrm{sat}_f(x),\ \mathrm{dep}_f(x, u) \succcurlyeq u$

$\forall v \in \mathrm{dep}_f(x, u)$: $v \succ u$ or $v = u$



Greedy Algorithm

Edmonds (1970)

Stable Allocation in Polymatroid Intersection

$(E, f, \succ_{\mathrm{F}}), (E, h, \succ_{\mathrm{H}})$: Ordered Polymatroids

$x \in \mathbf{P}(f) \cap \mathbf{P}(h)$:
Stable Allocation

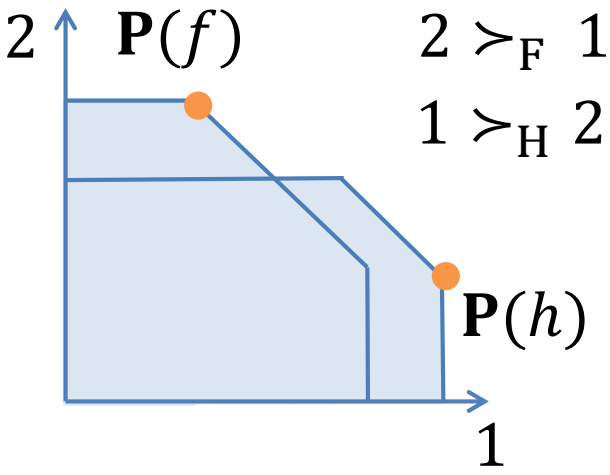
$\Leftrightarrow$ (△)

$\forall u \in E,$

$u \in \mathrm{sat}_f(x),\ \mathrm{dep}_f(x, u) \succcurlyeq_{\mathrm{F}} u$

or

$u \in \mathrm{sat}_h(x),\ \mathrm{dep}_h(x, u) \succcurlyeq_{\mathrm{H}} u$



Stable Allocation in Polymatroid Intersection

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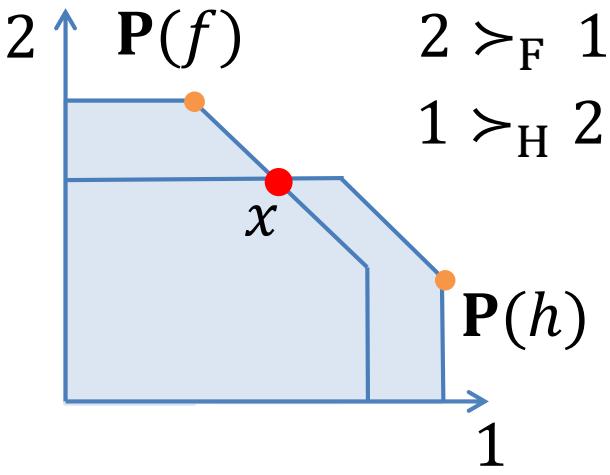
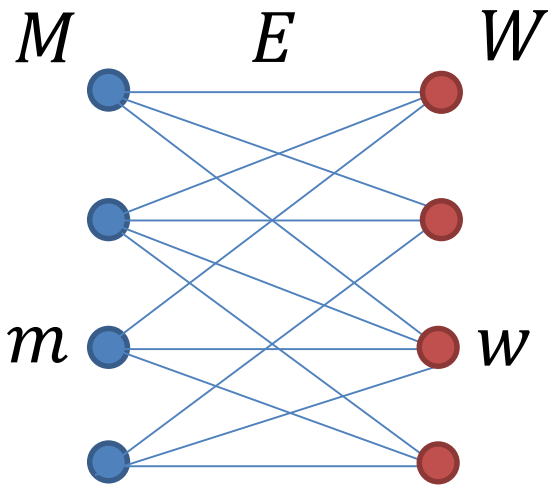


$2 \succ_{\mathrm{F}} 1$

$1 \succ_{\mathrm{H}} 2$

Example1: Stable Matching



$$h(A) \coloneqq |M \cap \partial A| \quad (A \subseteq E)$$

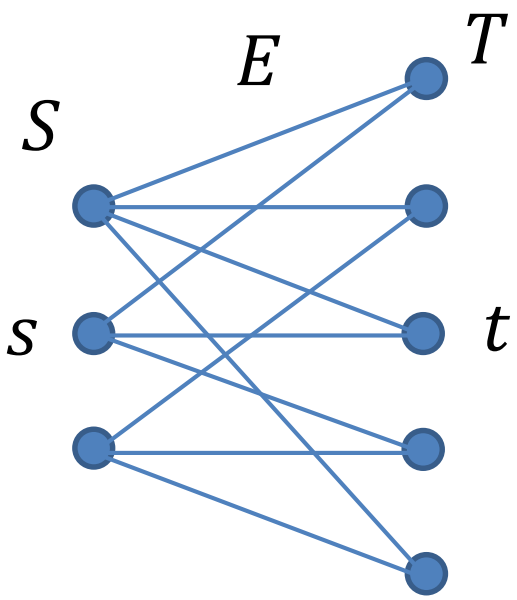
$$f(A) \coloneqq |W \cap \partial A| \quad (A \subseteq E)$$

$\succ_{\mathrm{H}}$: consistent with $\succ_m$ for each $m \in M$.

$\succ_{\mathrm{F}}$: consistent with $\succ_w$ for each $w \in W$.

Stable matching = Stable allocation of $(E, f, \succ_{\mathrm{F}}) \& (E, h, \succ_{\mathrm{H}})$

Example2: Task Assignment



How many hours each student $s \in S$ should be engaged in each task $t \in T$?

$$h(A) \coloneqq \sum_{s \in S} h_s(A \cap \delta s) \qquad (A \subseteq E)$$

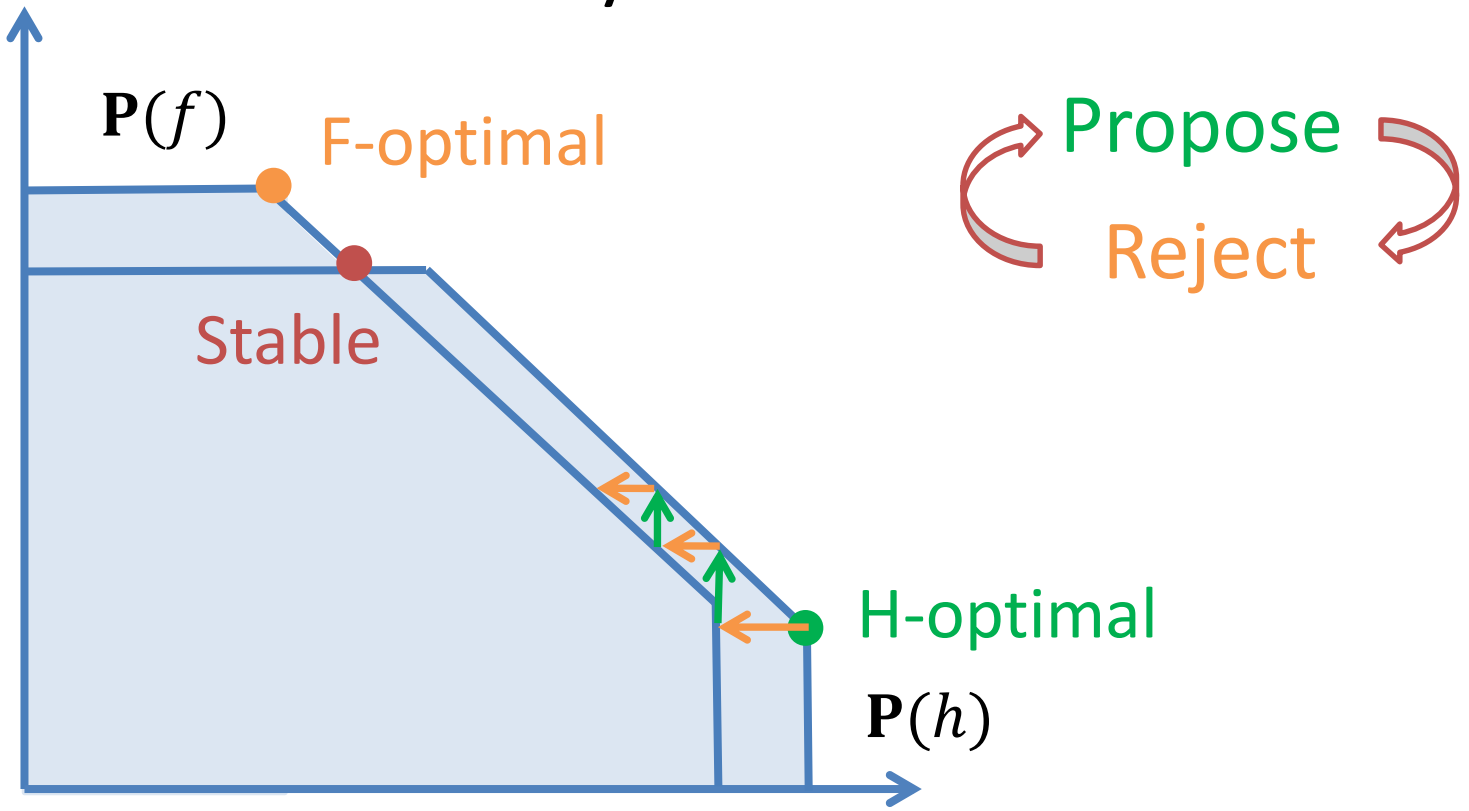
$$f(A) \coloneqq \sum_{t \in T} f_t(A \cap \delta t) \qquad (A \subseteq E)$$

$\succ_{\mathrm{H}}$: consistent with $\succ_s$ for each $s \in S$.

$\succ_{\mathrm{F}}$: consistent with $\succ_t$ for each $t \in T$.

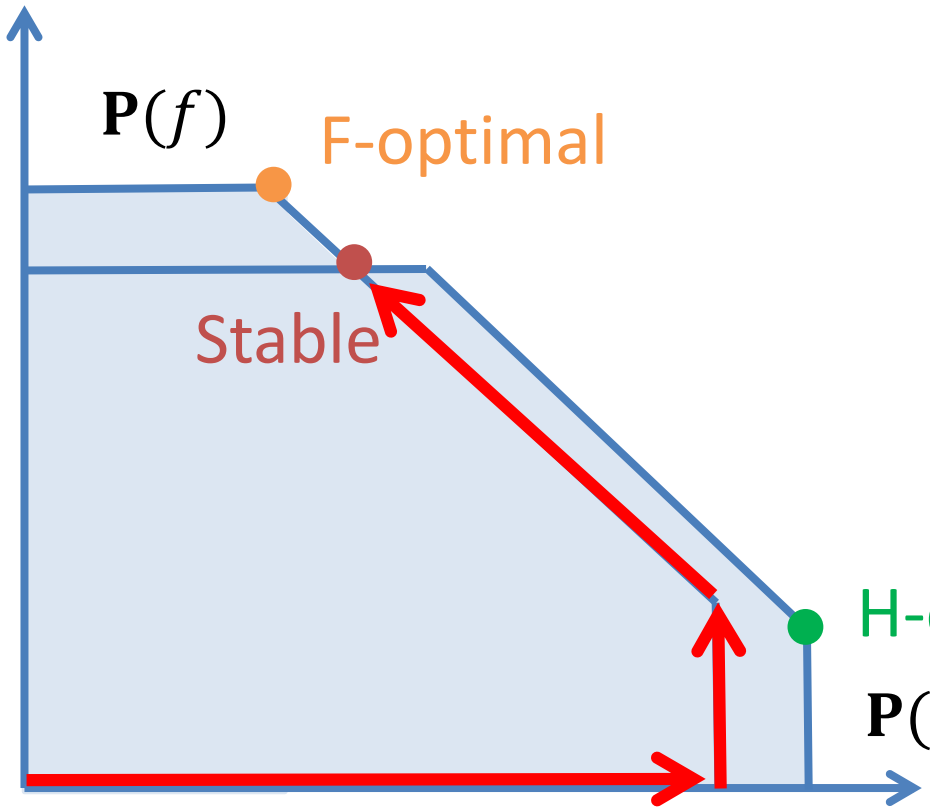
Extended Gale-Shapley Algorithm

— Why is it slow? —



Pseudo-polynomial time.☹

Our Algorithm



Exchange

Strongly polynomial time. ☺

Main Result

A Strongly Polynomial Algorithm for Finding a Stable Allocation in Polymatroid Intersection

Running Time Bound: $O(n^3\gamma)$

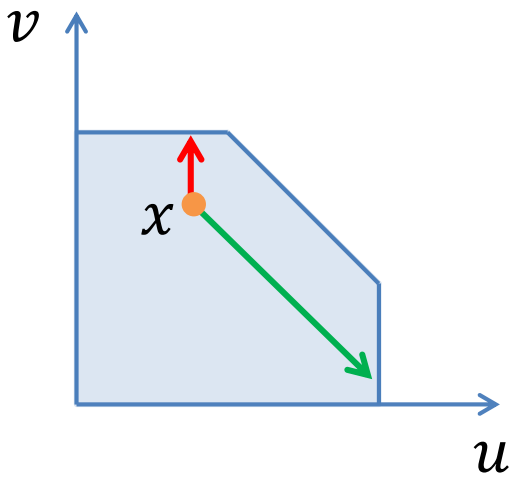
$n := |E|$

γ: Time for Computing
a Saturation/Exchange Capacity

Saturation/Exchange Capacity

(E, f): Polymatroid

$x \in \mathbf{P}(f)$



Saturation Capacity

$$\hat{c}_f(x, v) := \max\{\, \alpha \mid x + \alpha\chi_v \in \mathbf{P}(f) \,\}$$

Exchange Capacity

$$\bar{c}_f(x, u, v) := \max\{\, \alpha \mid x + \alpha(\chi_u - \chi_v) \in \mathbf{P}(f) \,\}$$

Algorithm Outline

- Set $x \coloneqq \mathbf{0}$, $R \coloneqq \emptyset$.
- Repeat

  Find an Augmenting Path/Cycle.

  Augment x along the Path/Cycle.

  $R \coloneqq R \cup \{\text{rejected elements}\}$

Algorithm Outline

- Set $x := \mathbf{0}$, $R := \emptyset$.
- Repeat

  Find an Augmenting Path/Cycle.

  Augment x a

  $R := R \cup \{$re

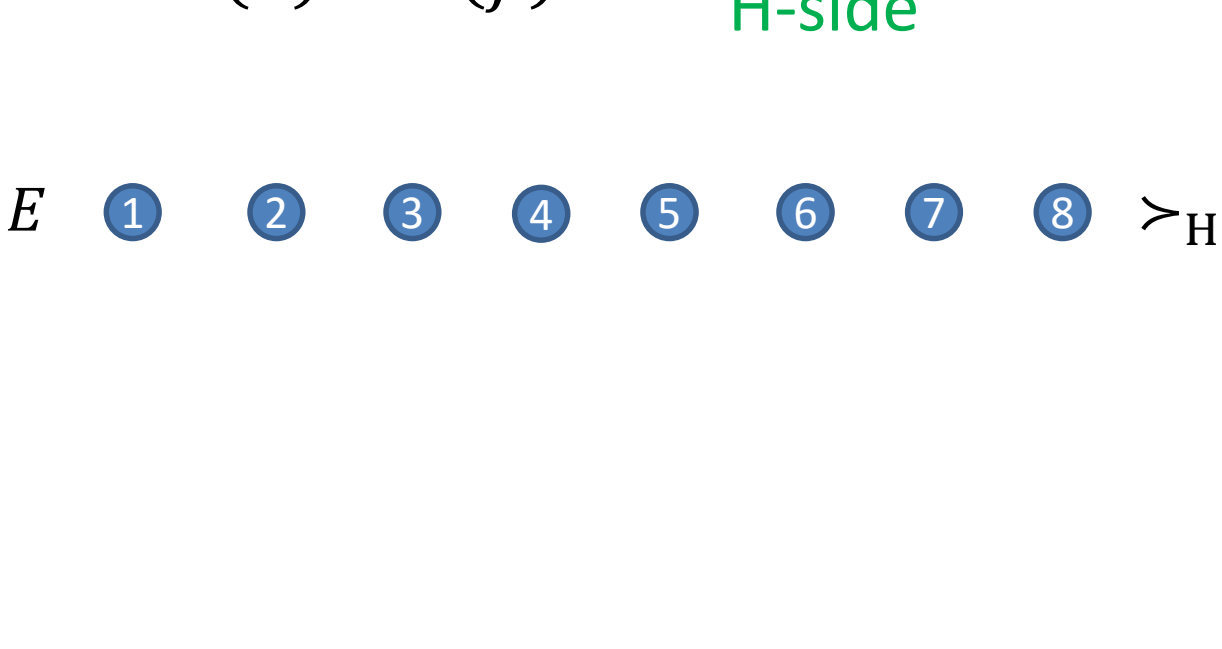
according to
the Policy of Gale-Shapley algo.

- Repeat propose & reject
- Once rejected, never proposed

cf. Polymatroid intersection algorithm
finds a shortest augmenting path

Auxiliary Graph $G(x)$

$x \in \mathbf{P}(h) \cap \mathbf{P}(f)$



- H-side: E arranged w.r.t. $\succ_{\mathrm{H}}$, exchangeability arcs w.r.t. h.

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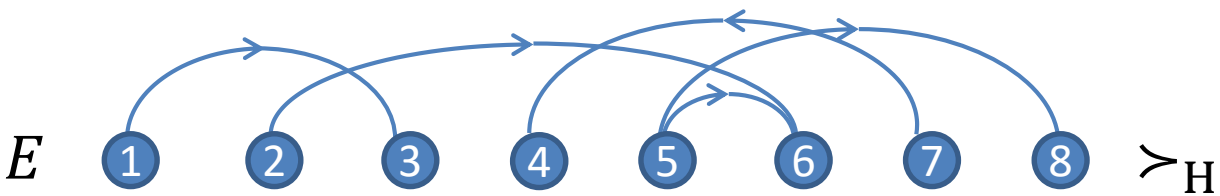

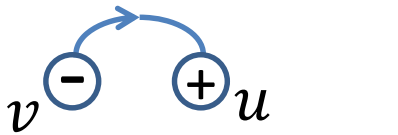

$\bar{c}_h(x, u, v) > 0$

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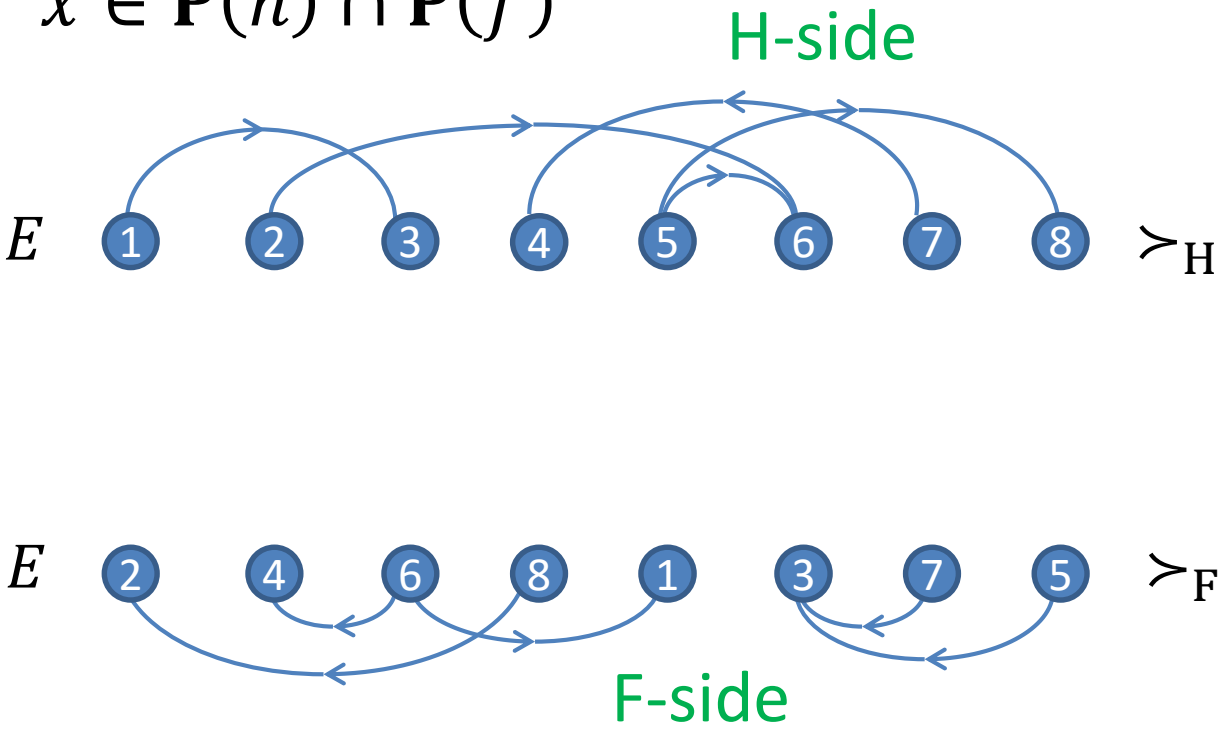

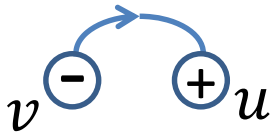

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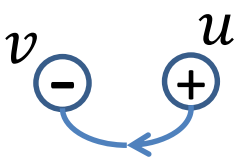

$\bar{c}_f(x,u,v) > 0$

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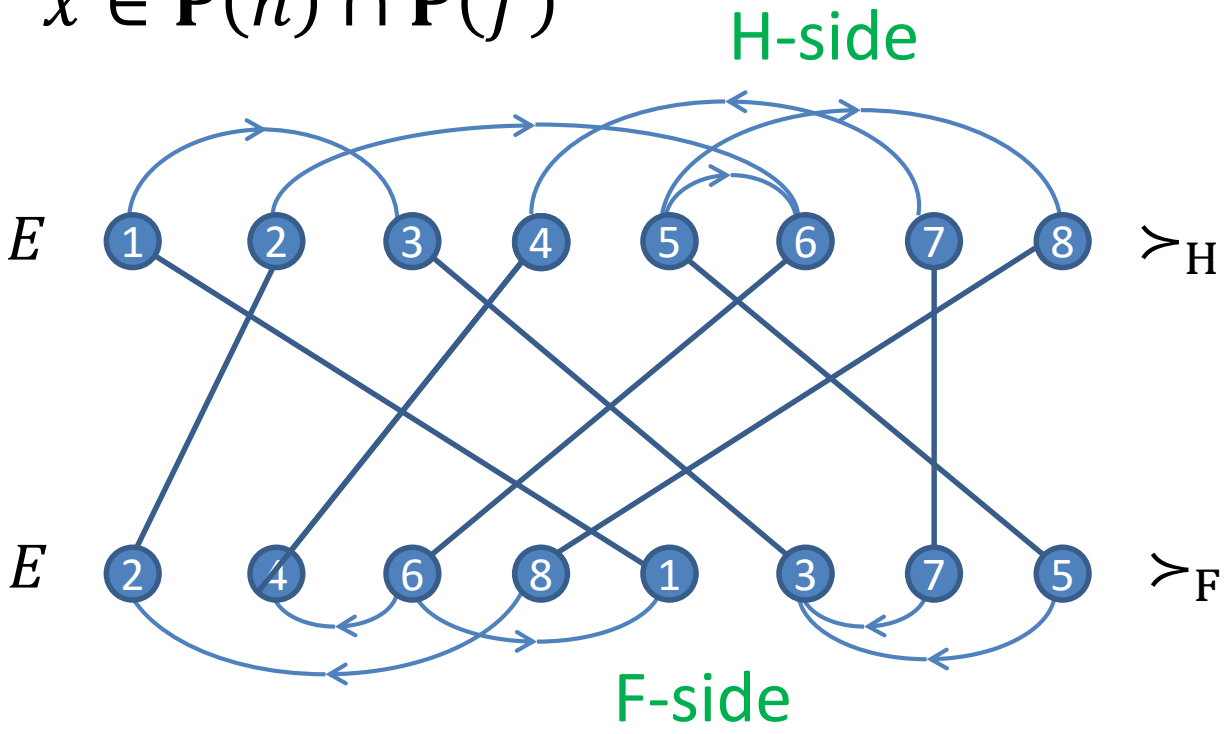

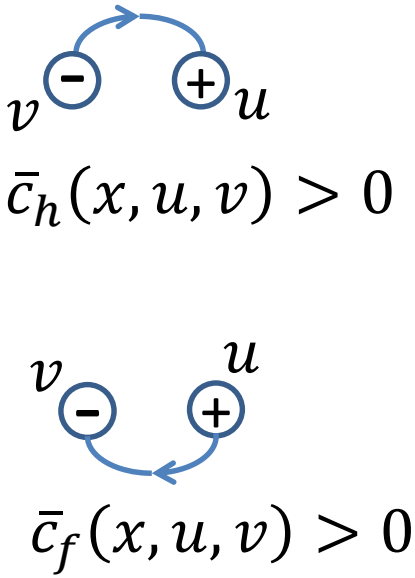

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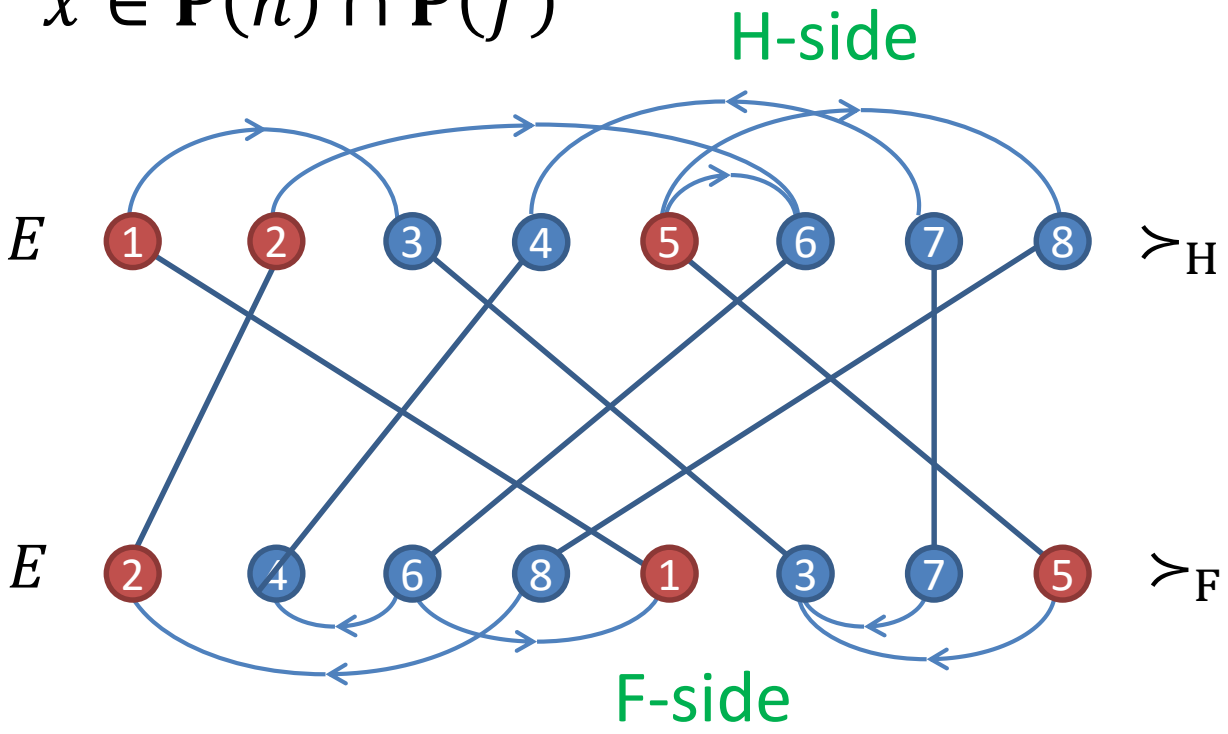

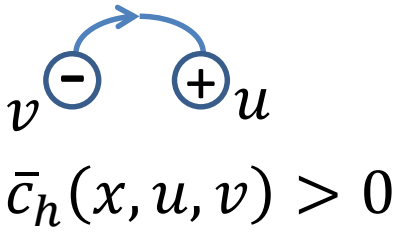

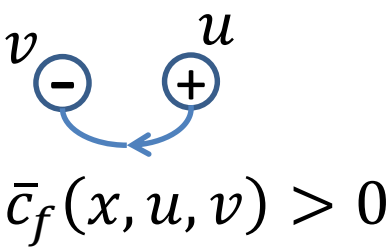

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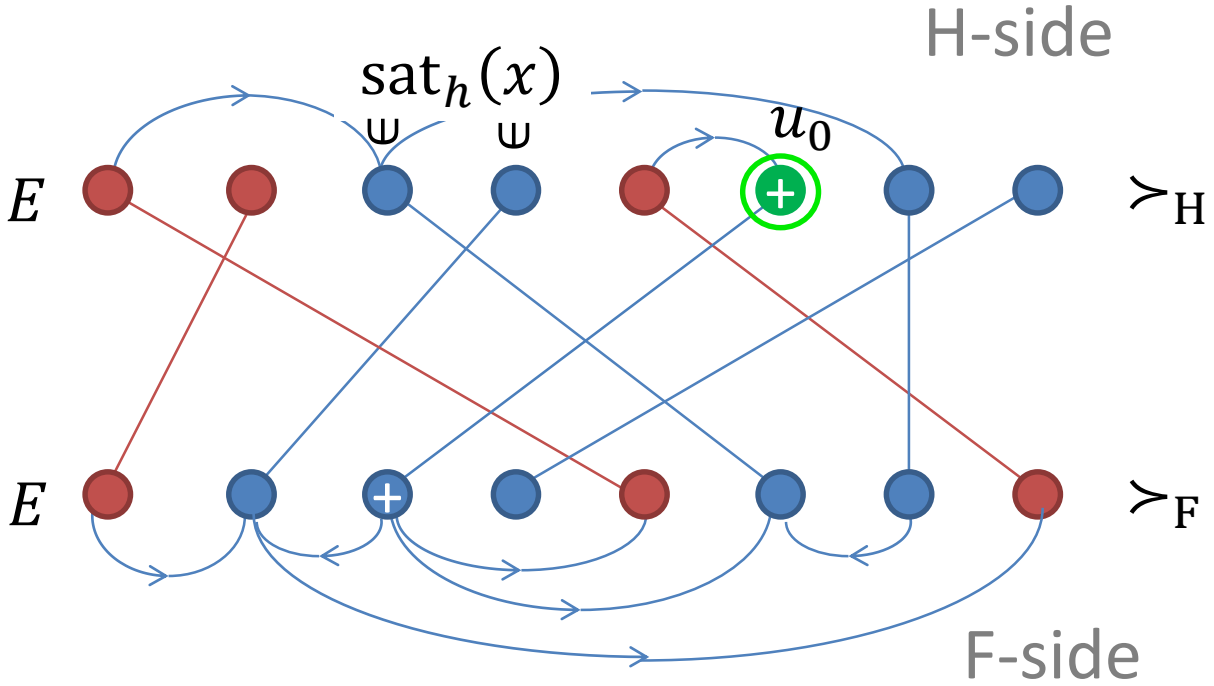
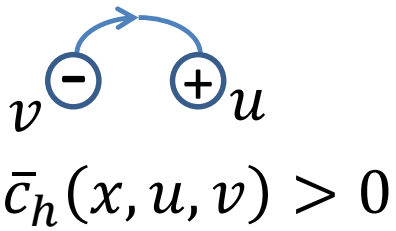
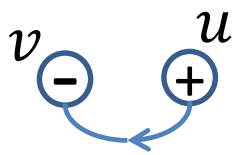
Search for Augmentation

1. $u_0 \coloneqq \max_{\succ_H} \{\, u \mid u \in E \setminus (\mathrm{sat}_h(x) \cup R) \,\}$ (propose)





$\bar{c}_h(x,u,v) > 0$



$\bar{c}_f(x,u,v) > 0$

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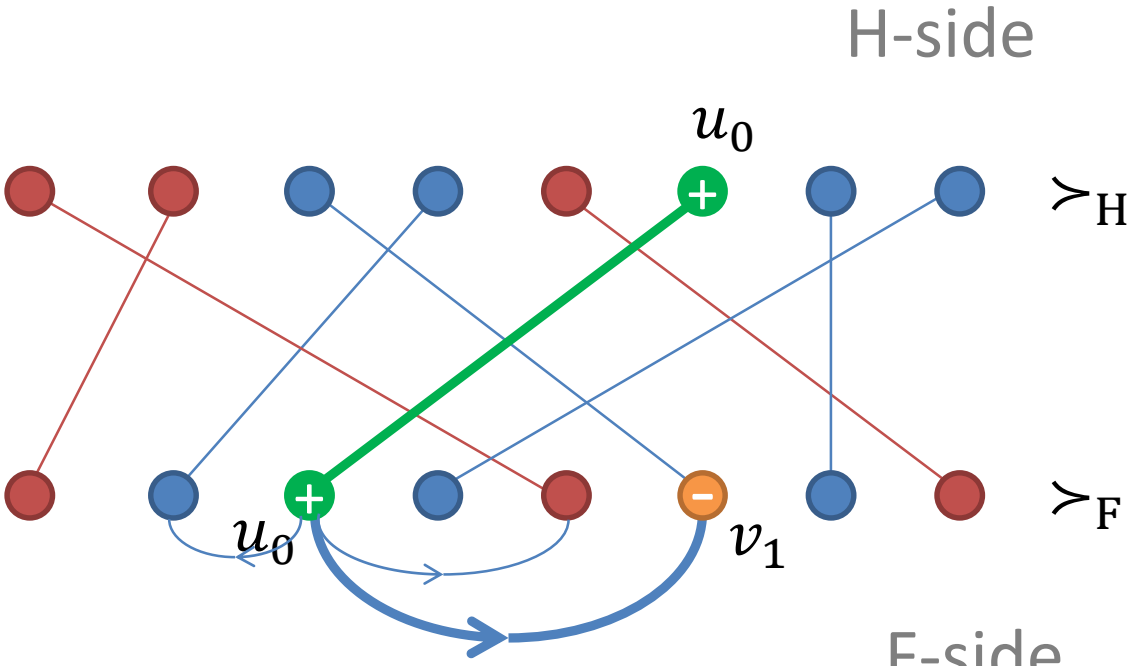
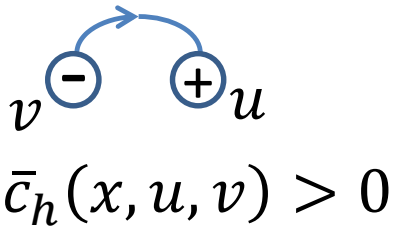
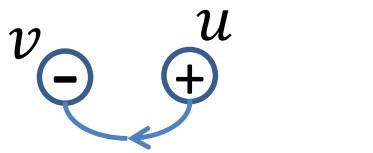
2. For $i = 1,2, \ldots$

   $v_i \coloneqq \min_{\succ_F} \{ v \mid \bar{c}_f(x, u_i, v) > 0 \}$ if $u_{i-1} \in \mathrm{sat}_f(x)$ (reject)







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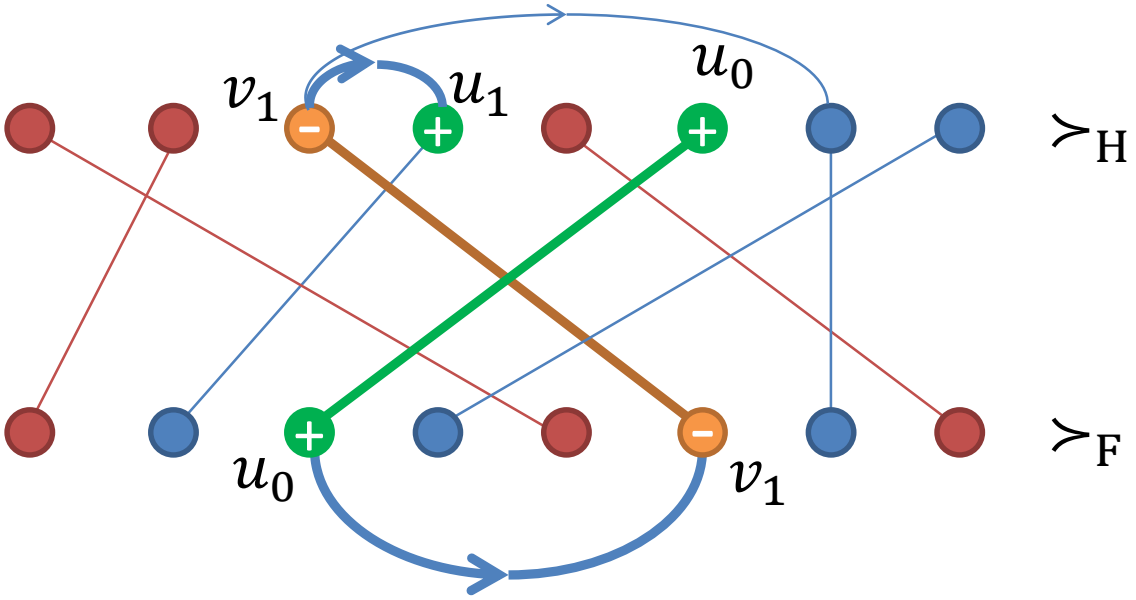
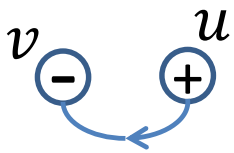
   $u_i \coloneqq \max_{\succ_H} \{\, u \mid u \in E \setminus (R + v_i),\ \ \bar{c}_h(x, u, v_i) > 0 \,\}$ (propose)

   until some path or cycle comes up.



$\bar{c}_h(x, u, v) > 0$



$\bar{c}_f(x, u, v) > 0$

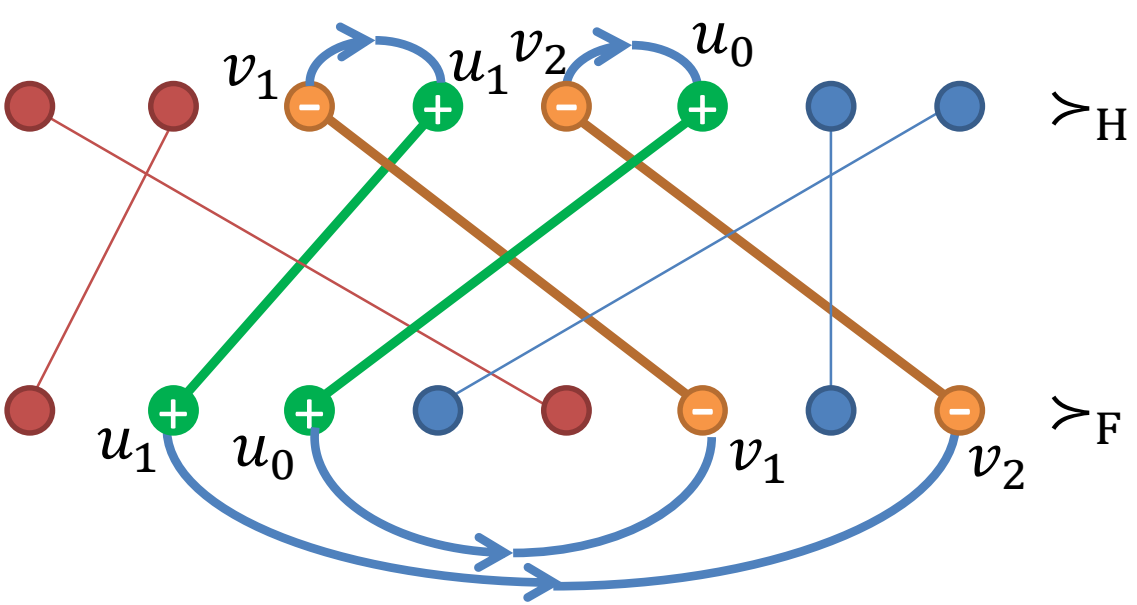
Search for Augmentation

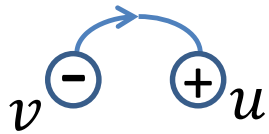
1. $u_0 \coloneqq \max_{\succ_H} \{\, u \mid u \in E \setminus (\mathrm{sat}_h(x) \cup R) \,\}$ (propose)
2. For $i = 1,2, \ldots$

   $v_i \coloneqq \min_{\succ_F} \{\, v \mid \bar{c}_f(x, u_i, v) > 0 \}$ if $u_{i-1} \in \mathrm{sat}_f(x)$ (reject)

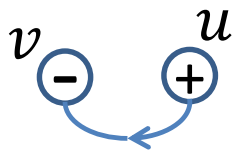
   $u_i \coloneqq \max_{\succ_H} \{\, u \mid u \in E \setminus (R + v_i),\ \ \bar{c}_h(x, u, v_i) > 0 \,\}$ (propose)

   until some path or cycle comes up.





$\bar{c}_h(x, u, v) > 0$



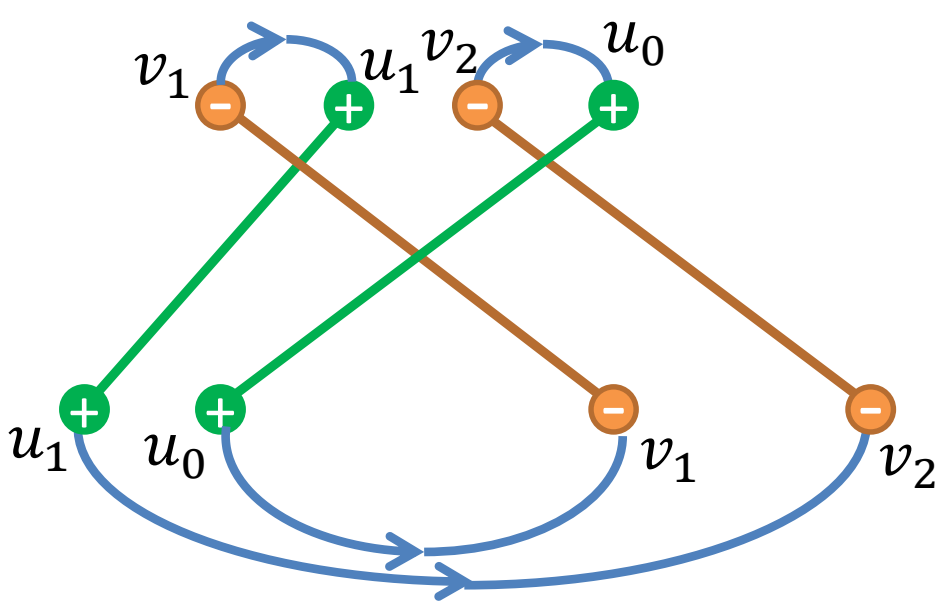
$\bar{c}_f(x, u, v) > 0$

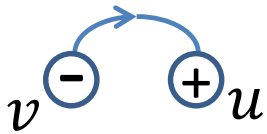
Augmentation

$$x := x + \alpha\left(\sum_j \chi_{u_j} - \sum_j \chi_{v_j}\right)$$

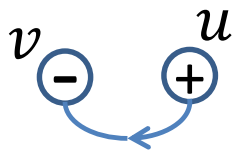
proposed elements (↑ χ_{u_j}) rejected elements (↑ χ_{v_j})

$$R := R \cup \{v_1, v_2, \dots\}$$





$\bar{c}_h(x, u, v) > 0$



$\bar{c}_f(x, u, v) > 0$

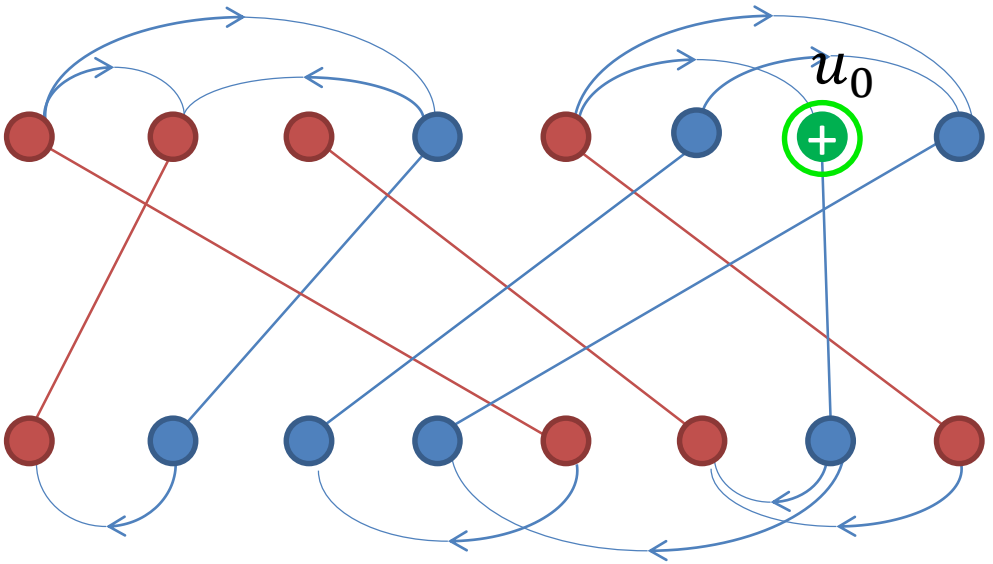
Search for Augmentation

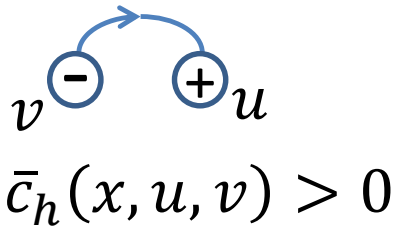
1. $u_0 \coloneqq \max_{\succ_H} \{\, u \mid u \in E \setminus (\mathrm{sat}_h(x) \cup R) \,\}$ (propose)
2. For $i = 1,2, \ldots$

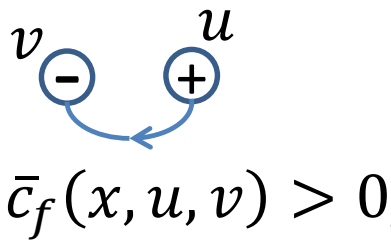
   $v_i \coloneqq \min_{\succ_F} \{\, v \mid \bar{c}_f(x, u_i, v) > 0 \}$ if $u_{i-1} \in \mathrm{sat}_f(x)$ (reject)

   $u_i \coloneqq \max_{\succ_H} \{\, u \mid u \in E \setminus (R + v_i),\ \ \bar{c}_h(x, u, v_i) > 0 \,\}$ (propose)

   until some path or cycle comes up.







Search for Augmentation

1. $u_0 \coloneqq \max_{\succ_H} \{ u \mid u \in E \setminus (\mathrm{sat}_h(x) \cup R) \}$ (propose)
2. For $i = 1,2, \ldots$

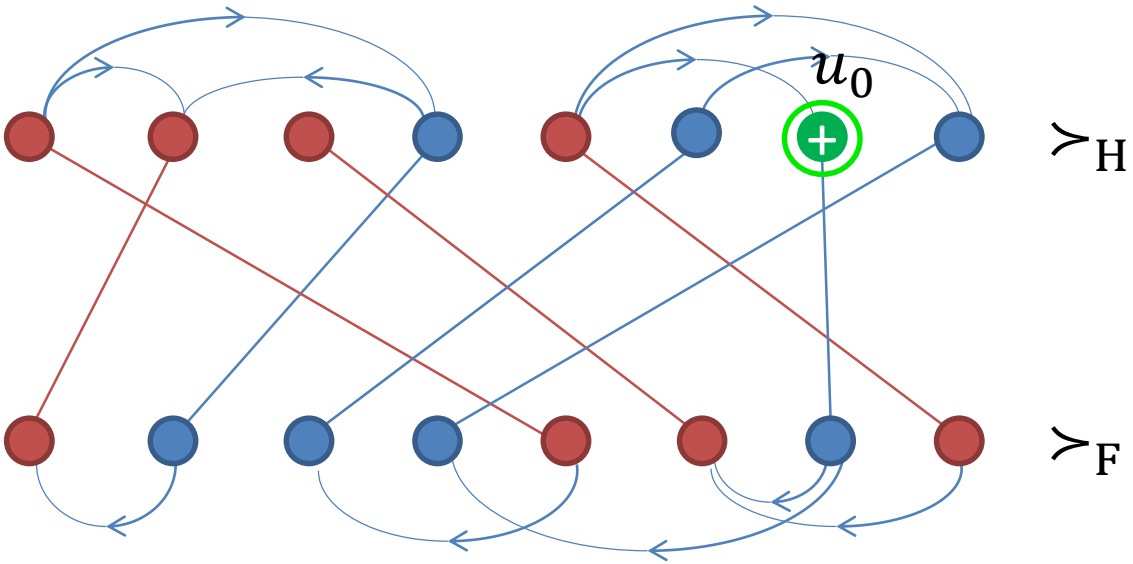
   $v_i \coloneqq$ r (reject)

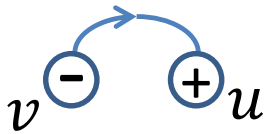
   $u_i \coloneqq$ r } (propose)

   until som

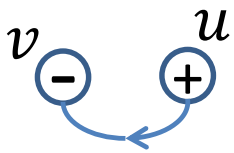
The algorithm terminates when

$$\mathrm{sat}_h(x) \cup R = E$$





$\bar{c}_h(x,u,v) > 0$



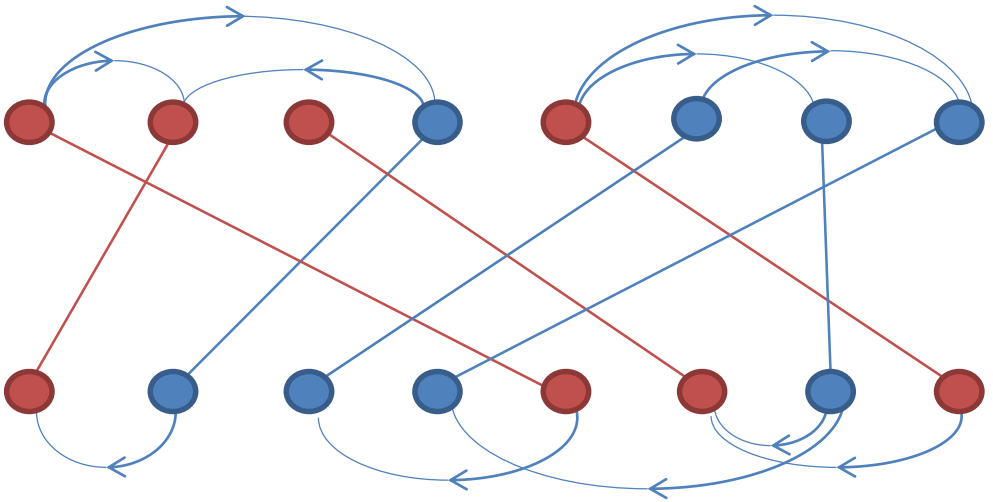
$\bar{c}_f(x,u,v) > 0$

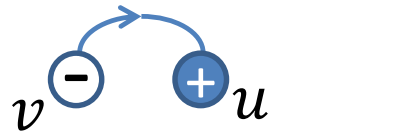
Invariants

- $x \in \mathbf{P}(f) \cap \mathbf{P}(h)$
- $\forall u \in \mathrm{sat}_h(x) \setminus R$:

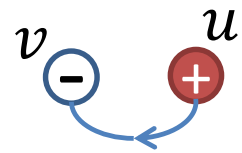
  $u \in \mathrm{sat}_h(x)$, $\forall v$: $[\, \bar{c}_h(x,u,v) > 0 \Rightarrow v \succcurlyeq_{\mathrm{H}} u]$.
- $\forall u \in R$:

  $u \in \mathrm{sat}_f(x)$, $\forall v$: $[\, \bar{c}_f(x,u,v) > 0 \Rightarrow v \succcurlyeq_{\mathrm{F}} u]$.





$\bar{c}_h(x,u,v) > 0$



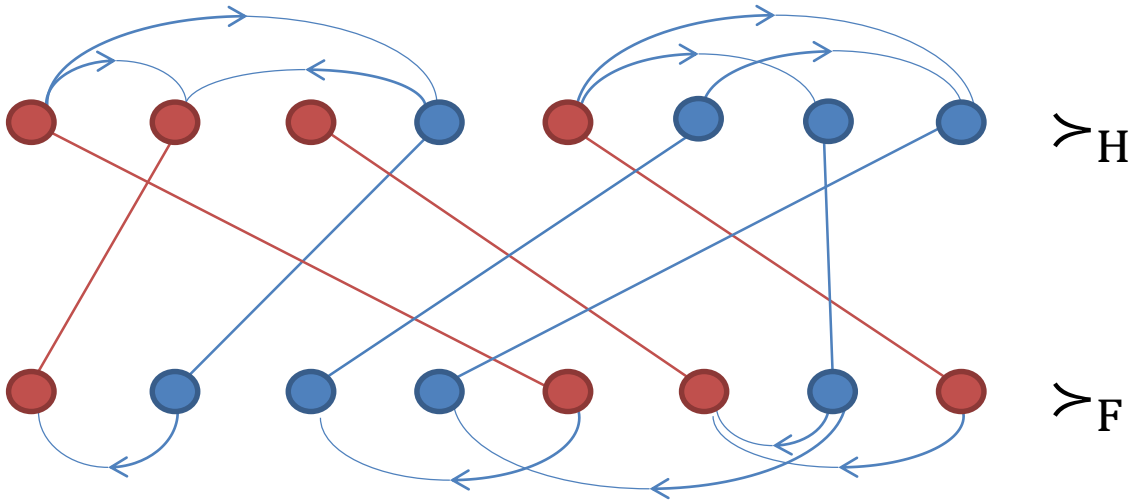
$\bar{c}_f(x,u,v) > 0$

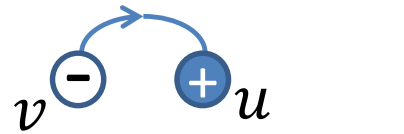
Invariants

- $x \in \mathbf{P}(f) \cap \mathbf{P}(h)$
- $\forall u \in \mathrm{sat}_h(x) \setminus R$:

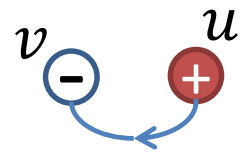
  $u \in \mathrm{sat}_h(x), \quad \mathrm{dep}_h(x,u) \succcurlyeq_{\mathrm{H}} u.$
- $\forall u \in R$:

  $u \in \mathrm{sat}_f(x), \quad \mathrm{dep}_f(x,u) \succcurlyeq_{\mathrm{F}} u$





$\bar{c}_h(x,u,v) > 0$



$\bar{c}_f(x,u,v) > 0$

Invariants

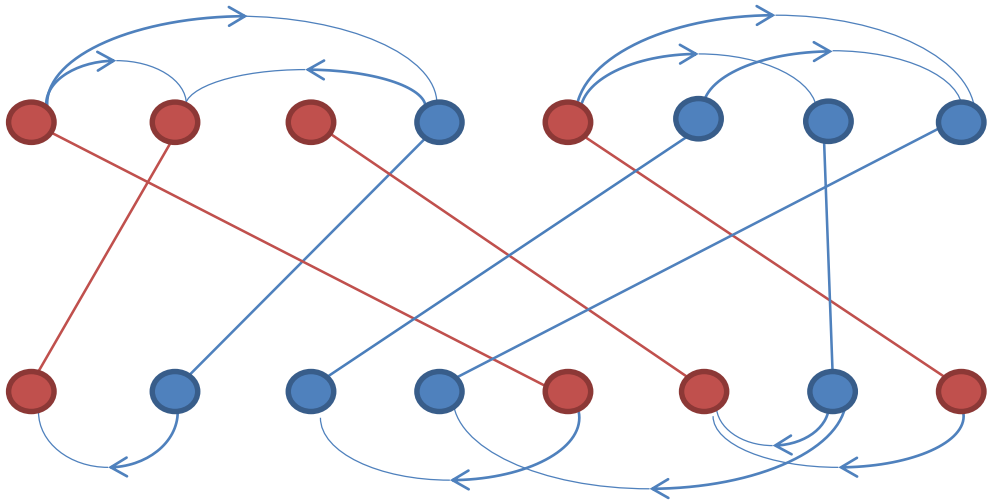
- $x \in \mathbf{P}(f) \cap \mathbf{P}(h)$
- $\forall u \in \mathrm{sat}_h(x) \setminus R$:

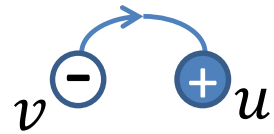
  $u \in \mathrm{sat}_h(x), \quad \mathrm{dep}_h(x,u) \succcurlyeq_{\mathrm{H}} u.$
- $\forall u \in R$:

  $u \in \mathrm{sat}_f(x), \quad \mathrm{dep}_f(x,u) \succcurlyeq_{\mathrm{F}} u$

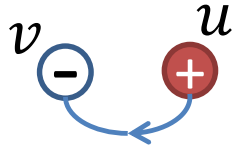
The algorithm terminates when

$$\mathrm{sat}_h(x) \cup R = E$$





$\bar{c}_h(x,u,v) > 0$



$\bar{c}_f(x,u,v) > 0$

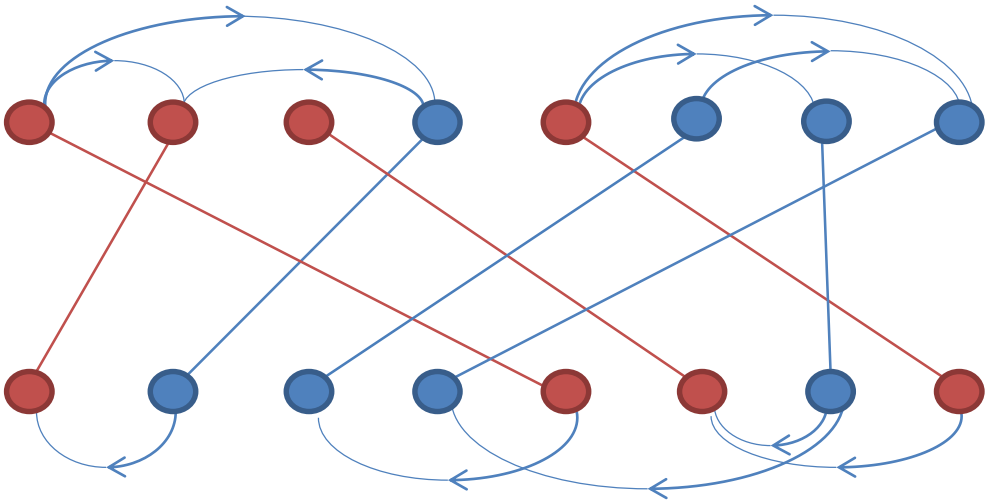
Goal

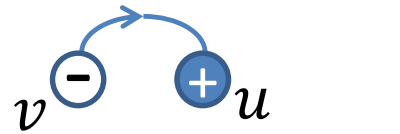
- $x \in \mathbf{P}(f) \cap \mathbf{P}(h)$

$\forall u \in E,$ $u \in \mathrm{sat}_h(x),\ \mathrm{dep}_h(x,u) \succcurlyeq_{\mathrm{H}} u$

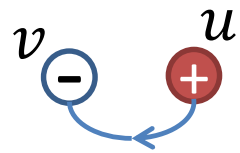
or

$u \in \mathrm{sat}_f(x),\ \mathrm{dep}_f(x,u) \succcurlyeq_{\mathrm{F}} u$





$\bar{c}_h(x,u,v) > 0$



$\bar{c}_f(x,u,v) > 0$

Running Time

- Each augmentation eliminates at least one exchangeability arc.
- That eliminated arc will never appear again.
- The total number of augmentation is $O(n^2)$.
- Finding an augmenting path/cycle requires $O(n\gamma)$ time.
- The overall running time is $O(n^3\gamma)$.

Questions

- The set of all the stable allocations forms a distributive lattice (shown by a reduction to a general model of Alkan and Gale (2003)). Does our algorithm find the H-optimal solution?

- Can one design a polynomial algorithm for finding a stable allocation in the discrete concave function model of Eguchi, Fujishige, and Tamura (2003)?