Mathieu moonshine and σ-models on K3

Roberto Volpato

Albert-Einstein-Institute
Max-Planck-Institut für Gravitationsphysik
Potsdam, Golm

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References

M.R.Gaberdiel, R.Volpato, 1206.5143 [hep-th]

M.R.Gaberdiel, S.Hohenegger, R.Volpato, 1106.4315, 1008.3778, 1006.0221 [hep-th]

Works in progress: M.R.Gaberdiel, D.Persson, H.Ronellenfitsch, R.Volpato

Monstrous Moonshine

Consider $PSL(2,\mathbb{Z})$-invariant J-function ($q = e^{2\pi i\tau}$):

$$J(\tau) = q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + \dots$$

Coefficients are dimension of reps of Monster group $\mathbb{M}$ [McKay]:

$$J(\tau) = \sum_{n=-1}^{\infty} q^n \dim V_n$$

$V_n \equiv$ representations of $\mathbb{M}$

McKay-Thompson series: $T_g(\tau) = \sum_{n=-1}^{\infty} q^n \, \mathrm{Tr}_{V_n}(g)$, for each $g \in \mathbb{M}$

$J(\tau)$ is the partition function of a chiral CFT $V^\natural$ ($c = 24$) with symmetry group $\mathbb{M}$

[Frenkel, Lepowski, Meurman '88]

$$J(\tau) = \mathrm{Tr}_{V^\natural}(q^{L_0 - \frac{c}{24}})$$

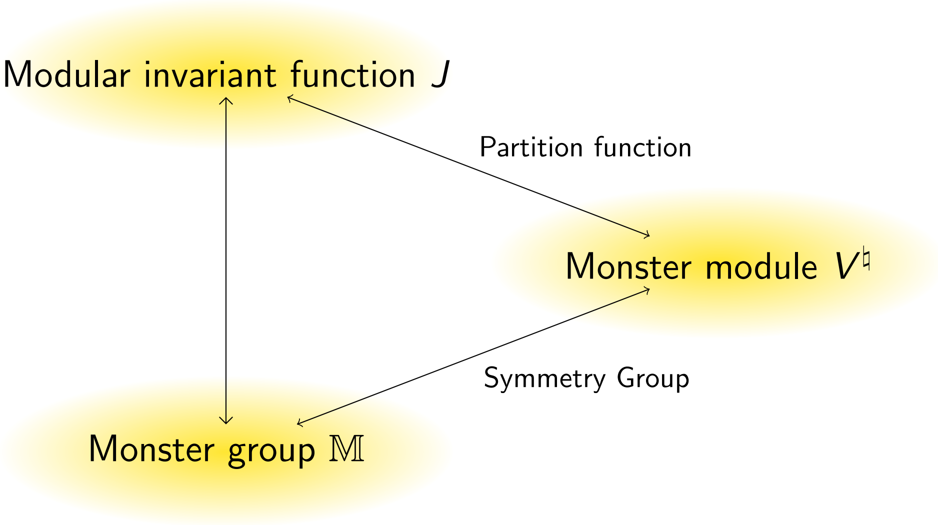
Monstrous Moonshine

Modular invariant function J

???

Monster group $\mathbb{M}$

Monstrous Moonshine

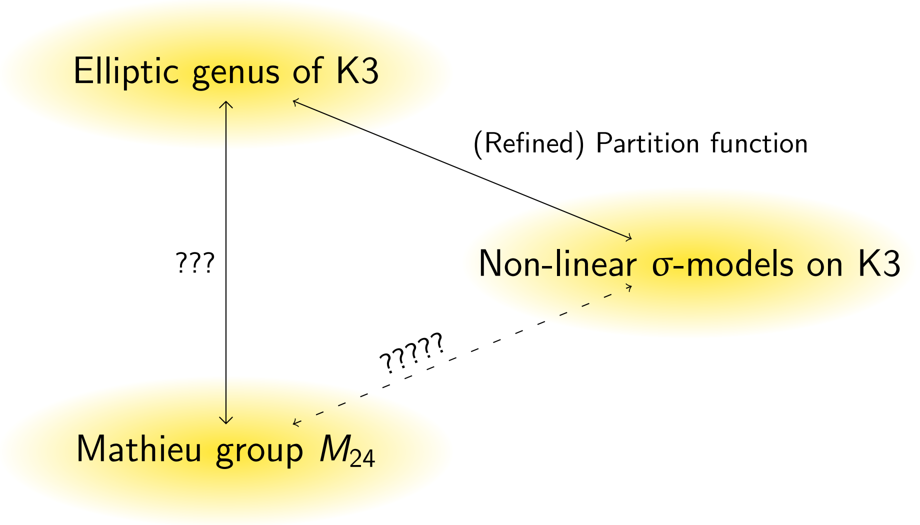


Mathieu Moonshine

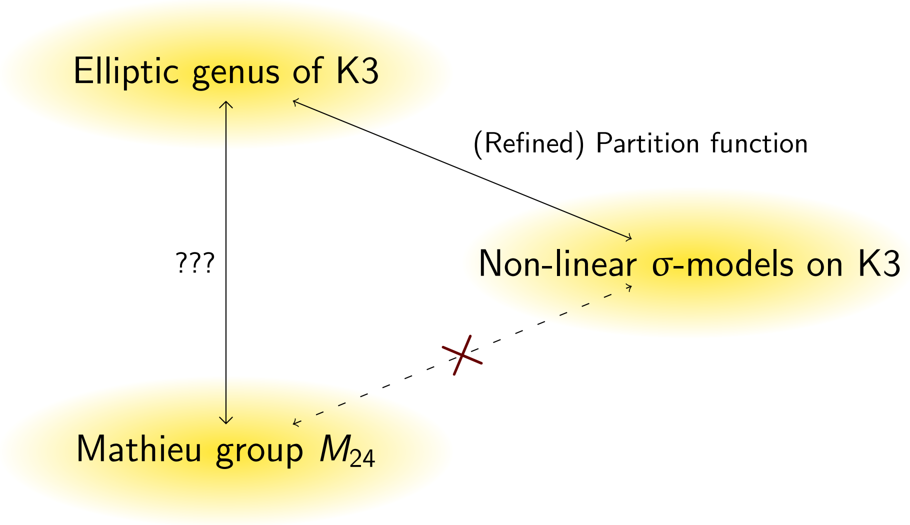
Elliptic genus of K3

↕ ???

Mathieu group M_{24}


Elliptic genus of K3
(Refined) Partition function
???
Non-linear σ-models on K3
?????
Mathieu group M_{24}

Mathieu Moonshine



Plan of the talk

1. Definitions and EOT conjecture
2. Twining genera
3. Symmetries of K3 models
4. Torus orbifolds and exceptional K3 models
5. Conclusions and open questions

The group M_{24}

M_{24} is a finite simple group of order

$$|M_{24}| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \ \sim\ 2 \times 10^8$$

Properties:

- Subgroup of S_{24}(=permutations of 24 symbols)
- Group of automorphisms of Niemeier lattice A_1^{24}/Weyl reflections
- 26 conjugacy classes
- 26 irreducible representations of dimensions

$$1, 23, 45, 231, 252, 253, 483, 770, 990, 1035, 1265,$$
$$1771, 2024, 2277, 3312, 3520, 5313, 5544, 5796, 10395$$

Elliptic genus: definition

- dim $= 2$ SuperCFT $\mathcal{N} = (4,4)$ with central charge $c = 6$
- Non-linear σ-models with target space K3
- The model depends on the choice of metric and B-field (moduli space of theories).

Elliptic genus

$$\phi_{K3}(\tau, z) = \mathrm{Tr}_{RR}\big((-1)^{F+\tilde{F}}\, q^{L_0 - \frac{c}{24}}\, \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}}\, y^{2J_0^3}\big)$$

where $q = e^{2\pi i \tau}$, $y = e^{2\pi i z}$.

J_0^3 is the 3rd comp of (left) $su(2)$ in $\mathcal{N} = (4,4)$ SC algebra

Elliptic genus: properties

$$\phi_{K3}(\tau,z)=\mathrm{Tr}_{RR}\Big((-1)^{F+\tilde F}\,q^{L_0-\frac{c}{24}}\,\bar q^{\tilde L_0-\frac{\tilde c}{24}}\,y^{2J_0^3}\Big)$$

- Independent of the moduli (metric and B-field)
- Holomorphic in τ and z
- Quasi-periodicity and modular properties:

$$\phi_{K3}(\tau,z+\ell\tau+\ell')=e^{-2\pi i(\ell^2\tau+2\ell z)}\,\phi(\tau,z)\qquad \ell,\ell'\in\mathbb{Z}$$

$$\phi_{K3}\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Big)=e^{2\pi i\frac{cz^2}{c\tau+d}}\,\phi(\tau,z)\qquad \begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{Z})$$

$\equiv$ (weak) Jacobi form of weight 0 and index 1

- $\phi_{K3}(\tau,z=0)=$ Euler number of K3 $=24$

- Distinct unitary RR reps of $\mathcal{N}=4$ at $c=6$ labeled by $(h,\ell)=$ Eigenvalues under (L_0, J_0^3) of the highest weight state

$$(h,\ell)=(\tfrac{1}{4},0)_{BPS};\quad (\tfrac{1}{4},\tfrac{1}{2})_{BPS};\quad (\tfrac{1}{4}+1,\tfrac{1}{2}),\ (\tfrac{1}{4}+2,\tfrac{1}{2}),\ldots$$

- Only states at $\bar{h}=\frac{1}{4}$ (*right*-moving ground states) contribute to ϕ_{K3}

Decomposition of ϕ_{K3} into *left* $\mathcal{N}=4$ characters at $c=6$

$$\phi_{K3}(\tau,z)=\sum_{(h,\ell)} A_{h,\ell}\ \mathrm{ch}_{h,\ell}(\tau,z)$$

where

- $\mathrm{ch}_{h,\ell}(\tau,z)=\mathrm{Tr}_{(h,\ell)}\big((-1)^{2J_0^3}\,q^{L_0-\frac{c}{24}}\,y^{2J_0^3}\big)$
- $A_{h,\ell}$ is (graded) multiplicity of (h,ℓ) rep

EOT observation

Multiplicities of (massive) $\mathcal{N} = 4$ reps:

$$\tfrac{1}{2}A_{h,\ell} = \quad 45, \quad 231, \quad 770, \quad 2277, \quad 5796, \quad 13915, \quad \ldots$$

Dimensions of irreps of M_{24}:

1, 23, 45, 231, 252, 253, 483, 770, 990, 1035, 1265,
1771, 2024, 2277, 3312, 3520, 5313, 5544, 5796, 10395

...and $13915 = 3520 + 10395$
(for higher h, many possible decompositions $\Rightarrow$ ambiguity)
[Eguchi, Ooguri, Tachikawa 1004.0956]

Conjecture: There is an action of M_{24} on the space of states contributing to the K3 elliptic genus

A Mathieu Moonshine?

$A_{h,\ell}$ are the dimensions of reps $\mathcal{R}_{h,\ell}$ of Mathieu group M_{24}

$$\phi_{K3}(\tau, z) = \sum_{(h,\ell)} \dim \mathcal{R}_{h,\ell} \; \mathrm{ch}^{\mathcal{N}=4}_{h,\ell}(\tau, z)$$

We can define the *twining genera*

[Cheng 1005.5415; Gaberdiel, Hohenegger, R.V. 1006.0221]

$$\begin{aligned}\phi_g(\tau, z) &= \mathrm{Tr}_{RR}\big(g\,(-1)^{F+\tilde{F}} q^{L_0-\frac{c}{24}}\,\bar{q}^{\tilde{L}_0-\frac{\tilde{c}}{24}}\,y^{2J_0^3}\big), \qquad g \in M_{24}\\ &= \sum_{(h,\ell)} \mathrm{Tr}_{\mathcal{R}_{h,\ell}}(g)\;\mathrm{ch}^{\mathcal{N}=4}_{h,\ell}(\tau, z)\end{aligned}$$

In Monster Moonshine, the analogous objects are McKay-Thompson series

A Mathieu Moonshine?

If g commutes with $\mathcal{N} = (4,4)$ SConf symmetry

$$\downarrow$$

ϕ_g must be a weak Jacobi form under ($N = ord(g)$)

$$\Gamma_0(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}) \mid c \equiv 0 \mod N \right\} \subset SL(2,\mathbb{Z})$$

We should also allow for a non-trivial multiplier system

[Gaberdiel, Hohenegger, R.V. 1008.3778]:

$$\phi_g\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = \chi_g\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) e^{2\pi i \frac{cz^2}{c\tau+d}} \phi_g(\tau, z) \qquad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$$

where $\chi_g : \Gamma_0(N) \to U(1)$

(Similar to what happens with McKay-Thompson series)

Results

For each M_{24} conjugacy class g there is a function $\phi_g(\tau, z)$ such that

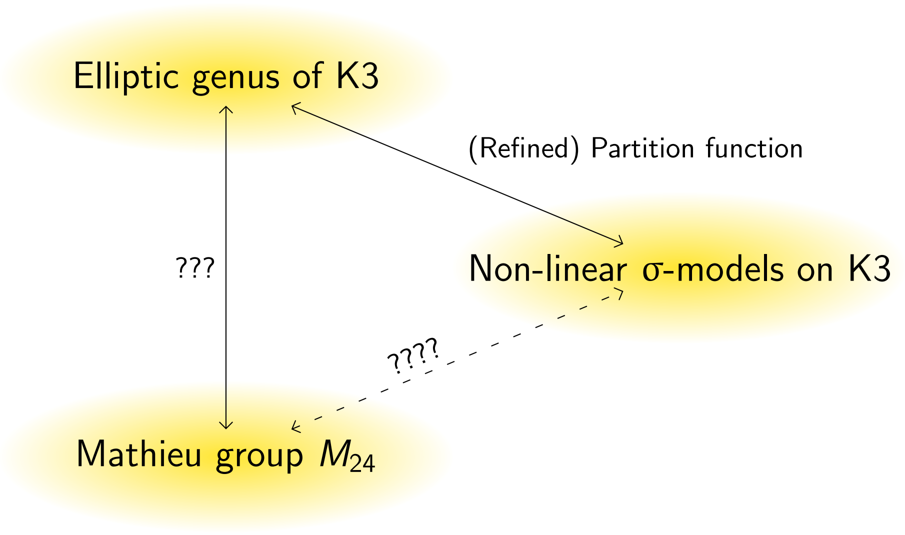
1. $\phi_e(\tau, z)$ ($e \equiv$ identity in M_{24}) is the elliptic genus of K3
2. ϕ_g is a weak Jacobi form for $\Gamma_0(N)$, $N =$ order of g
3. There are non-trivial M_{24}-representations $\mathcal{R}_{h,\ell}$

$$\phi_g(\tau, z) = \sum_{(h,\ell)} \mathrm{Tr}_{\mathcal{R}_{h,\ell}}(g) \ \mathrm{ch}^{\mathcal{N}=4}_{h,\ell}(\tau, z)$$

The representations $\mathcal{R}_{h,\ell}$ with these properties are unique
(but the surprising fact is *existence* rather than uniqueness...)

[Gaberdiel, Hohenegger, R.V. 1008.3778; Eguchi, Hikami 1008.4924; Gannon]

Mathieu Moonshine



A relation between K3 and Mathieu groups

A theorem in algebraic geometry gives a connection between K3 and Mathieu groups

Mukai Theorem

Finite groups of symplectic automorphisms of K3 surf's are subgroups of $M_{23} \subseteq M_{24}$.

[Mukai 1988; Kondo 1998]

- A symplectic automorphism of the K3 target space induces a symmetry of the corresponding σ-model.
- But σ-models may have non-geometrical (quantum) symmetries

Can we classify the groups of discrete symmetries of K3 σ-models?
(Quantum analogue of Mukai theorem)

Classification Theorem

- Let Co_0 be the group of automorphisms of the Leech lattice Λ (even self-dual lattice of dim 24 with no elements of norm 2)
- Let G be a group of symmetries of a K3 σ-model that commute with $\mathcal{N} = (4,4)$ and spectral flow.

Theorem

G is a subgroup of $Co_0 \equiv Aut(\Lambda)$ fixing pointwise a sublattice of Λ of rank at least 4.

Conversely, any $G \subset Co_0$ fixing a sublattice of Λ of rank at least 4 is the symmetry group of some K3 model.

[Gaberdiel, Hohenegger, R.V. 1106.4315]

Also M_{24} is a subgroup of Co_0, but...

Subgroups $G \subset Co_0$ fixing a sublattice of rank 4:

- $G \subset \mathbb{Z}_2^{12} \rtimes M_{24}$ (at least 4 orbits on $\mathbf{24}$-dim rep)
- $G = 5^{1+2}.\mathbb{Z}_4$
- $G = \mathbb{Z}_3^4 \rtimes A_6$ or $G = 3^{1+4}.\mathbb{Z}_2.G''$ with $G'' = 1, \mathbb{Z}_2$ or $\mathbb{Z}_2^2$

What can we learn from this description?

- There is no G such that $M_{24} \subseteq G$
- There are some G such that $G \not\subset M_{24}$

Sketch of the proof

Known facts about σ-models on K3:
[Aspinwall 9611375, Nahm, Wendland 9912067]

- There are 24 RR ground states at $h = \tilde{h} = \frac{1}{4}$ ($\equiv \mathbb{R}^{4,20}$)
- The ground states are contained in $\mathcal{N} = (4,4)$ supermultiplets:
  - 4 are in one $(h, \ell; \bar{h}, \bar{\ell}) = (\frac{1}{4}, \frac{1}{2}; \frac{1}{4}, \frac{1}{2})$ (subspace $\Pi \subset \mathbb{R}^{4,20}$)
  - 20 are in 20 distinct $(h, \ell; \bar{h}, \bar{\ell}) = (\frac{1}{4}, 0; \frac{1}{4}, 0)$
- The D-brane charges form an even self-dual lattice $\Gamma^{4,20}$
- We can think of $\Gamma^{4,20}$ as embedded in (the dual of) $\mathbb{R}^{4,20}$

Moduli space of K3 models:

$$O(\Gamma^{4,20}) \backslash O(4,20,\mathbb{R}) / (O(4,\mathbb{R}) \times O(20,\mathbb{R}))$$

Sketch of the proof

G: group of symmetries that commute with $\mathcal{N} = (4,4)$ and spectral flow

1. G acts faithfully on the lattice $\Gamma^{4,20} \subset \mathbb{R}^{4,20}$
2. G fixes pointwise $\Pi \subset \mathbb{R}^{4,20}$, with $\dim \Pi = 4$
3. G acts faithfully on the sublattice $L = \Gamma^{4,20} \cap \Pi^\perp$ with $\dim L \leq 20$
4. L can be embedded into the Leech Λ and the action of G extends to automorphisms of Λ
5. The sublattice $\Lambda^G \subset \Lambda$ of vectors fixed by G is the orthogonal complement of $L \subset \Lambda$

$$\downarrow$$

G is a subgroup of $Aut(\Lambda) \equiv Co_0$ that fixes a sublattice of rank≥ 4

Discussion

Problems:

1. There are no K3 σ-models with symmetry group $\mathbb{M}_{24}$
2. There are groups of symmetries $G \not\subset \mathbb{M}_{24}$ (exceptional cases)
3. All groups are contained in Co_0 but no Conway Moonshine

Open questions:

- Are there unknown SCFTs with the same elliptic genus?
- Is $\mathbb{M}_{24}$ the symmetry of some different 'structure' related to K3?
- Are exceptional models in some sense 'special'?

Exceptional models seem related to orbifolds of non-linear σ models on T^4
[Gaberdiel, R.V. 1206.5143]

Cyclic torus orbifolds

CFT orbifold Given a CFT $\mathcal{C}$ with a symmetry g, project on g-invariant states and introduce new (twisted) sectors

(Cyclic) Torus orbifolds

Special families of K3 models are given by orbifolds (in CFT sense) of non-linear σ-models on T^4 by some $\mathbb{Z}_n$ group of symmetries

- Easiest example: non-linear σ-model with target space $T^4/\mathbb{Z}_N$, where $\mathbb{Z}_N$ is a group of automorphisms of T^4
- In general, group $\mathbb{Z}_N$ might be non-geometric symmetry (e.g. asymmetric orbifolds)

Results I

Classification of orbifolds of T^4 models:

- Symmetric (geometric) orbifolds with $ord(g) = 2, 3, 4, 6$ (known)
- Asymmetric orbifolds with $ord(g) = 4, 5, 6, 8, 10, 12$ (NEW!)

Explicitly constructed $\tilde{\mathcal{C}}/\tilde{g}$ for $ord(\tilde{g}) = 5$

[Gaberdiel, R.V. 1206.5143]

Results II

Results

1. All torus orbifolds are exceptional
2. *Most* exceptional models are torus orbifolds.

(a) $G \subset \mathbb{Z}_2^{12} \rtimes M_{24}$ (at least 4 orbits on $\mathbf{24}$-dim rep)

(b) $G = 5^{1+2}.\mathbb{Z}_4$

(c) $G = \mathbb{Z}_3^4 \rtimes A_6$ or $G = 3^{1+4}.\mathbb{Z}_2.G''$ with $G'' = 1, \mathbb{Z}_2$ or $\mathbb{Z}_2^2$

In particular:

- case (b) $\Leftrightarrow$ $\mathbb{Z}_5$ orbifold of T^4 model
- case (c) $\Leftrightarrow$ $\mathbb{Z}_3$ orbifold of T^4 model

[Gaberdiel, R.V. 1206.5143]

Conclusions and open questions

- Strong evidence of a Mathieu moonshine
- Interpretation in terms of non-linear σ-model is problematic
- More and more evidence that there is a CFT behind it: even the *Generalised Moonshine* works! (see Daniel's talk)

Many open questions:

- Analog of genus zero property of Montrous Moonshine? [Cheng, Duncan]
- Relation with the old M_{24} η-product Moonshine? [Mason]
- Relation with the *Umbral Moonshine*? [Cheng, Duncan, Harvey]
- M_{24}-moonshine in 'second quantized' string on K3?