UNIVERSAL GRAPHS WITH FORBIDDEN SUBGRAPHS

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(JOINT WORK WITH SHELAH)

Abstract

Problem (1). *For which connected finite graphs C is there a countable universal C-free graph?*

Problem ($1'$). *For which connected finite graphs C is there a countable universal C-free graph with oligomorphic automorphism group? (I.e., there are finitely many orbits in the action on n-tuples, for each n.)*

We expect a complete solution to both Problems to emerge eventually.

Problem $1'$ is a necessary step toward Problem 1, and a much clearer problem in practice. There is a clear combinatorial criterion for problem $1'$, and where we have satisfactory partial results on Problem 1, they come by analyzing some borderline cases of Problem $1'$.

When there is an oligomorphic automorphism group then we may look at it as a topological group with interesting topological dynamics, following [KPT 2005].

One really should allow not just a single forbidden subgraph C, but a finite set of forbidden graphs $\mathcal{C}$. Then an explicit solution is not expected, but it is reasonable to ask whether there is a finite algorithm which answers the question.

Some concrete results are as follows.

Theorem 1. *Let $\mathcal{C}$ (or C, if $\mathcal{C} = \{C\}$), be one of the following. Then there is a countable universal $\mathcal{C}$-free graph.*

- $\mathcal{C} = \emptyset$ [Rado 1964] *(oligomorphic)*
- C *complete* [Henson 1971] *(oligomorophic)*
- C *a path or nearpath* [KMP1988, ChT 2007] *(oligomorphic: paths only)*
- $\mathcal{C}$ *is homomorphism-closed* [ChShSh 1999] *(oligomorphic)*
- C *is a bow tie (two triangles joined at a vertex)* [Komjáth 1999] *(oligomorphic)*

Theorem 2. *Conversely, in the following cases, if there is a universal $\mathcal{C}$-free graph then it is on the list above.*

- C *is 2-connected* [FK 1997]
- C *is a tree* [ChShe 2005]
- $\mathcal{C}$ *is a set of cycles* [ChShi 1996]

For the case of a single forbidden subgraph, the results in [FK 1997] suggest that C shoud be made of solid blocks.

I will suggest an approach toward a proof of this. The main tools are the method of [FK 1997] and some ideas of Shelah that give an inductive approach.

I may also say something about the evidence for and against an algorithmic solution in general, and the topic of metric universality, specifically the metrically homogeneous graphs considered by Moss and Cameron.

References

[ChShe 2005] G. Cherlin and S. Shelah, Universal graphs with a forbidden subtree. Preprint, Fall 2005.

[ChShSh 1999] G. Cherlin, S. Shelah, and N. Shi, Universal graphs with forbidden subgraphs and algebraic closure Advances in Applied Mathematics **22** (1999), 454–491.

[ChShi 1996] G. Cherlin and N. Shi, Graphs omitting a finite set of cycles. *J. Graph Theory* **21** (1996), 351–355.

[ChT 2007] G. Cherlin and L. Tallgren, Universal graphs with a forbidden near-path or 2-bouquet, *J. Graph Theory* **56** (2007), 41–63.

[ER 1963] P. Erdős and A. Rényi. Asymmetric Graphs, *Acta Math. Acad. Sci. Hungar.* **14** (1963), 295–315.

[FK 1997] Z. Füredi and P. Komjáth, On the existence of countable universal graphs. *J. Graph Theory* **25** (1997), 53–58.

[Henson 1971] C. Ward Henson, A family of countable homogeneous graphs. *Pacific J. Math.* 38:69–83, 1971.

[KPT 2005] A. Kechris, V. Pestov, S. Todorčevič. Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups, *Geom. Funct. Anal.* **15** (2005), 106–189

[Komjáth 1999] P. Komjáth, Some remarks on universal graphs. *Discrete Math.* **199** (1999), 259–265.

[KMP1988] P. Komjáth, A. Mekler and J. Pach, Some universal graphs, *Israel J. Math.* **64** (1988), 158–168.

[Rado 1964] R. Rado, Universal graphs and universal functions. *Acta Arith.* **9** (1964), 331–340.

[Moss 1989] L. Moss Existence and nonexistence of universal graphs. *Fund. Math.* **133** (1989), 25–37.

[Moss 1991] L. Moss The universal graphs of fixed finite diameter. *in* Graph theory, combinatorics, and applications.Vol. 2 (Kalamazoo, MI, 1988), 923937, Wiley-Intersci. Publ., Wiley, New York, 1991.

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