We are interested in topological dynamics for subgroups of $S_\infty$ (automorphism groups).

Our main motivation will be the connections with Ramsey theory.

**History**

The field started with the work of Pestov.

* Pestov (’98): $\text{Aut}(\mathbb{Q}, <)$ is extremely amenable.

* Glasner - Weiss (’02): they calculated the universal minimal flow of $S_\infty$.
  It was the first interesting explicit calculation.

* Glasner - Weiss (’03): universal minimal flow of $\text{Homeo}(2^\mathbb{N})$.

All these results were proved using Ramsey theory.

Kechris - Pestov - Todorcevic (’05): In the general framework of Fraïssé limits, they showed that there is a precise correspondence between the Ramsey property for the age of the structure and the extreme amenability of the automorphism group.

The obstructions to having the Ramsey property are encoded in the universal minimal flow.
From this, new Ramsey theorems were proved, new extremely amenable groups were found, new universal minimal flows were computed.

But very often, the applications go this way:

Ramsey $\rightarrow$ extremely amenable.

In principle, the correspondence goes both ways. Here, we discuss a possible approach to go the other direction.

Rk: This is not absurd; several theorems like van den Waerden's theorem, Hindman's theorem, have nice dynamical proofs.

A few names: Graham - Rothschild, Nešetřil - Rödl, Szamen, Nguyen van Thé ...

Framework

We look at closed subgroups of $S_n$. These are known as non-archimedean Polish groups: their defining property is to have a basis at $1_0$, consisting of open subgroups.

* It is a known fact that every such group is $\text{Aut}(M)$ for some homogeneous $M$, that is, $M$ is the Fraïssé limit of some class.

* Since all this works for all such groups, in particular for discrete groups, it makes sense to impose a condition: oligomorphicity.
G N M is oligomorphic iff there are only finitely many substructures of M of size n up to isomorphism.

Re: Since M is homogeneous, this is the same as oligomorphic in the usual sense.

**Dynamics**

For now, it will be very general. We consider continuous actions \( G \times X \) on compact Hausdorff spaces: \( G \)-flows.

These form a category: maps are \( G \)-morphisms, that is, continuous \( G \)-maps (maps which commute with the action of \( G \)).

**Def:**

- \( X \) is called **topologically transitive** if there exists a dense orbit in \( X \).
- \( X \) is called **minimal** if every orbit is dense. This means that there are no proper subflows.
- If we fix a point \( x_0 \) in \( X \) with a dense orbit, \((X, x_0)\) is called an **ambit**. This is a \( G \)-flow.

Ambit morphisms are \( G \)-morphisms which send the distinguished point to the distinguished point. Now, morphisms are unique.

**Def:**

\[ \text{RUCB}(G) = \{ f: G \to C \mid \| f \|_{\infty} < \infty \text{ and } \forall \varepsilon > 0, \exists \delta > 0, \forall g \in G, \forall x \in U, \| f(g) - f(g') \| < \varepsilon \}. \]
These are very relevant functions to look at:

if \((X, x_0)\) is an ambit and \(f \in C(X)\), then we can lift \(f\) to \(\tilde{f} \in RUCB(G)\) by putting \(\tilde{f}(g) = f(g \cdot x_0)\).

\(^*\) And \(RUCB(G)\) forms an algebra which is equipped with an action of \(G\):

\[(g, f)(h) = f(g^{-1} \cdot h).\]

This action is continuous.

\text{Rk: This is not true of the action of \(G\) on \(C_b(G)\).}

\(^*\) This algebra is universal in the following sense.

Let \(S(G)\) be the maximal ideal space of \(RUCB(G)\).

Then \(S(G)\) is a compactification of \(G\) and \(RUCB(G) = C(S(G))\).

\(G\) embeds in \(S(G)\) so \(1_G\) is a distinguished point of \(S(G)\).

\(\to (S(G), 1_G)\) is the universal ambit of \(G\):

for every ambit \((X, x_0)\), there is a unique \(G\)-map \(\pi : S(G) \rightarrow X\):

\[1_G \mapsto x_0\]

\(^*\) Usually, \(S(G)\) is a very big thing.

\text{ex: If} \(G\) \text{ is discrete, } RUCB(G) = l^\infty(G) \text{ and } S(G) = \beta G, \text{ the space of ultrafilters on } G.

\text{And this is not metrizable unless } G \text{ is compact.}
This universal object $S(G)$ carries more structure: it has a semi-group structure.

If $p, q \in S(G)$, there exists a (unique)
$G$-map $\pi : S(G) \to S(G)$;
\[ 1_G \mapsto q \]
we then define $p \cdot q = \pi(p)$.

Another way of seeing this is the following. The group $G$ is dense in $S(G)$ so there exists a net $g_x$ of elements of $G$ converging to $p$. Then $p \cdot q$ is the limit of $g_x \cdot q$.

Exercise: This law is associative and extends the group operation.

The semi-group is right-topological:
the map $p \mapsto pq$ (which is $\pi$) is continuous for all $q \in S(G)$.

Rk: Multiplication on the left is not necessarily continuous.

As a semi-group, $S(G)$ has left-ideals:
subsets $I$ of $S(G)$ such that $S(G)I \subseteq I$.

Rk: Since the semi-group is right-topological, the interesting ideals to look at are left-ideals.

We have the correspondence:
closed left-ideals $\leftrightarrow$ sub-flows of $S(G)$
minimal left-ideals $\leftrightarrow$ minimal sub-flows.
And minimal ideals exist by Zorn's lemma.
Let M be a minimal left-ideal.
Then M is universal for minimal flows of G: if X is a minimal G-flow and \( x_0 \in X \) is arbitrary, then by universality of S(G), there exists a continuous G-map

\[
\pi : (S(G), 1_G) \rightarrow (X, x_0).
\]

By minimality, we then have \( \pi(M) = X \).

The main fact about these minimal ideals is that universal minimal flows is the following.

Th: Ellis

M is coalescent, i.e. if \( \pi : M \rightarrow M \) is a homeomorphism of G-flows, then \( \pi \) is actually an automorphism.

Why is it important?

It implies that the universal minimal flow is unique up to isomorphism.

Indeed, if \( M' \) is another universal minimal flow, the universality gives two surjective maps \( M \rightarrow M' \). Their compositions are homeomorphisms, thus automorphisms by coalescence.

Proof: The most fundamental lemma is the following.

Lemma: Ellis

If S is a compact right-topological semigroup, then S contains an idempotent \( (e^2 = e) \)
Proof (of the lemma): Zorn's lemma.

[Exercise]

Rk: Hindman's theorem follows directly from this lemma and is definitely non-trivial.

- First observation: every idempotent $e \in M$ is a right identity: for all $p \in M$, $pe = p$.

$Me$ is a left-ideal (because $M$ was). And since $M$ is minimal, $Me = M$.

So if $p \in M$, there is $q \in M$, s.t. $qe = p$.

Then we just calculate:

$$pe = qee = qe = p.$$

- Now let's start with our endomorphism

$\pi: M \rightarrow M$.

It commutes with the $G$-action so in particular, we have:

$$\forall p \in S, \forall x \in M, \pi(px) = p\pi(x). \ (\ast)$$

(A priori, $\pi$ only commutes with the group action, but $G$ is dense and right-multiplication is continuous so we get the equality above.)

Fix some idempotent $e \in M$ and set $p = \pi(e)$.

We will see that $\pi$ is nothing but right-multiplication by $p$:

$$\pi(x) = \pi(xe) = x\pi(e) = xp.$$
Again, $M_p$ is a left ideal, so by minimality, $M_p = M$.
This means that there is $q \in M$ s.t. $qq = e$.
Let $\rho : M \to M$ be the right multiplication $x \mapsto xq$
by $q$. This is again a morphism of $G$-flows (by associativity).
We have
$(\pi \circ \rho)(x) = \pi(xq) = xe = x$,
so $\pi \circ \rho$ is the identity.
And since both maps are surjective (all
homomorphisms are surjective, by minimality),
this implies that $\pi$ is an automorphism.

$\square$

Remark: The proof is simple and algebraic, but the theorem is a very powerful tool.

Veech: If $G$ is locally compact non compact, then $M(G)$ is not metrizable.
But what is surprising is that when we look at bigger groups, $M(G)$ can be metrizable.
It can even be a point.

Definition: $G$ is extremely amenable if $M(G)$ is a point.
(The terminology was introduced by Granier.)
Now let's do something which is particular to subgroups of $S_\infty$.

**Note:** Let $G$ be a closed subgroup of $S_\infty$. Then $S(G)$, and also $M(G)$, are $0$-dimensional.

**Proof:** Let $V$ be an open subgroup of $G$. We consider the map 
\[ l^\infty(V \setminus G) \to RUCB(G) \]
\[ f \mapsto \tilde{f} \]
defined by $\tilde{f}(g) = f(Vg)$. (Because $G \leq S_\infty$, $\tilde{f}$ is in $RUCB(G)$.) It is easy to check that $\bigcup_{V \in G, \text{open}} l^\infty(V \setminus G)$ is dense in $RUCB(G)$.

Moreover, functions in $l^\infty(V \setminus G)$ that take only finitely many values are dense (choose an $E$-dense finite set in the interval).

And continuous functions which take only finitely many values are locally constant. Their density means that they separate points and thus, that $S(G)$ is totally disconnected, hence $0$-dimensional.

**Rk:** Again, this is very particular to subgroups of $S_\infty$. It is sufficient to understand $0$-dimensional flows.
We want to prove the following theorem of Pestov which was mentioned in the introduction.

\[ \text{Th: Pestov} \]
\[ G = \text{Aut} (\mathbb{Q}, <) \text{ is extremely amenable.} \]

We begin with a general fact:

**Observation:** To prove that \( G \) is extremely amenable, it suffices to prove that every minimal subset of \( \mathbb{Q}^k \) is trivial.

Stabilizer of the finite tuple \( \vec{a} \).

**Proof (of the observation):** Suppose that \( M(G) \) is non-trivial: \( \exists x_0, x_1 \in M \text{ with } x_0 \neq x_1 \).

Since \( M(G) \) is 0-dimensional, we can find a clopen set \( D \subseteq M \) that separates them.

Now \( D \) is clopen so \( G_D = \{ g \in G \mid g.D = D \} \) is an open subgroup of \( G \), which means there exists \( \vec{a} \in \mathbb{Q}^k \) s.t. \( G_{\vec{a}} \leq G_D \).

Define \( \pi : M \to \mathbb{Q}^k_{\vec{a}} \) by
\[
\pi(x)(G_{\vec{a}} g) = 1 \iff g \cdot x \in D.
\]

This is a \( G \)-map.

And we have
\[
\begin{cases}
\pi(x_0)(1_e) = 0 \quad \text{(D separates } x_0 \text{ from } x_1) \\
\pi(x_1)(1_e) = 1.
\end{cases}
\]

Thus \( \pi(M) \) is minimal and non-trivial too.

\( \square \)

**Rk:** The actions on right and left cosets are isomorphic.
Proof (of the theorem): We show that if \( z_0 \in 2^{G/\alpha} \), then \( \overline{G \cdot z_0} \) contains a fixed point.

Since the action of \( G \) on \( G/\alpha \) is transitive, the only points that can be fixed are the constants \( \overline{0} \) and \( \overline{1} \).

\[ \overline{0} \in \overline{G \cdot z_0} \iff \exists \text{finite } F \subseteq G, \exists g \in G \text{ s.t.} \]
\[ z_0^{-1} F \cdot G \alpha = \overline{0}. \]

We can reformulate the problem so that the result becomes a consequence of the Ramsey theorem: \( G/\alpha \) can be identified with ordered tuples the size of \( \alpha \) (because elements of \( G/\alpha \) are determined by where they send \( \overline{a} \)).

**Rk:** \( G/\alpha = \{ \text{all embeddings } \overline{a} \rightarrow M \} \)

and the Ramsey theorem speaks about \((M, \alpha)\).

In order to identify \( G/\alpha \) and \((M, \alpha)\), we need some kind of rigidity so we add a linear ordering.

Then \( z_0 \) is a coloring of tuples of \( \alpha \).

Now the Ramsey theorem gives an arbitrarily large set in which all tuples have the same color, say \( \overline{0} \). Then starting with \( F \) (which corresponds to a finite set of tuples), we can translate \( F \) to this large set, and get that \( \overline{0} \in \overline{G \cdot z_0} \).

**Rk:** Outside \( S_0 \), a good analogue of \( 2^{G/\alpha} \) should be the action of \( G \) on Lipschitz functions on \( G \).
Precompact homogeneous spaces

Let $H$ be a closed subgroup of $G$. We equip $G/H$ with the following uniformity:

$g_1 H = g_2 H \iff \exists u \in U, \forall g \in H, g u H = g_2 H.$

(*$U$*)

This uniformity is countably generated so it is metrizable. It is compatible with the topology of $G/H$.

Usually, it is not complete.

**Def:** $G/H$ is precompact if the completion of this uniformity is compact, i.e., if $V \exists U \subseteq G$ finite s.t. $U F H = G$.

When does this appear? With permutation groups.

**Fact:** Let $G$ act on a structure $M$ and $H$ be a closed subgroup of $G$.

Then $G/H$ is precompact iff for all $\bar{a} \in M^k$, the $G$-orbit $G \cdot \bar{a}$ splits in finitely many $H$-orbits.

**Rk:** If $G \cdot M$ is oligomorphic, then $G/H$ is precompact iff $H \cdot M$ is oligomorphic.

This happens very often.

**Proof (of the fact):** $\iff$ Let $V$ be an open subgroup of $G$.

Then there exists $\bar{a} \in M^k$ s.t. $G \cdot \bar{a} \subseteq V$.

The orbit $G \cdot \bar{a}$ splits as $H \cdot (F \cdot \bar{a})$ where $F \subseteq G$ is finite.
Now we check that $HFV = G$.

Rk: It is the same condition as with $UFH$ (passing to inverses).

Let $g \in G$.
Then there are $f \in F$ and $h \in H$ s.t. $g \cdot \overline{a} = hf \cdot \overline{a}$, 
so $f^{-1}h^{-1}g \in G \overline{a} \leq V$ and $g \in HFV$.

$\Rightarrow$ If an orbit $C \cdot \overline{a}$ is given, there is a finite set $F \subseteq G$ s.t. $HFG \overline{a} = G$.
Then $C \cdot \overline{a} = H \cdot (F \cdot \overline{a})$.

* What is this compactification $\hat{G}/H$?

Let $G = \text{Aut}(M)$, with $M$ w-categorical and homogeneous.
Let $H = \text{Aut}(M^*)$, with $M^*$ also w-categorical and homogeneous.

Then $G/H$ is precompact ($H \cap M$ is oligomorphic) and $\hat{G}/H$ can be viewed as the set of all expansions of $M$ in the language of $M^*$ that satisfy the universal theory of $M^*$, i.e. the theory of $M^*$ obtained when using only $\forall$ quantifiers.

Ex: $G = S_{\infty}$
$H = \text{Aut}(\mathbb{Q}, <)$
Then $\hat{G}/H = \text{LO}$.

$G = S_{\infty}$
$H = \text{Aut}(\text{random graph})$.
Then $\hat{G}/H = \text{all graphs on the original structure } M$. 
This object $\hat{G}/H$ has the following universality property: if $GA(X,x_0)$ is an ambit such that $H \cdot x_0 = x_0$, then there is a unique morphism $\hat{G}/H \rightarrow X$.

\[ H \mapsto x_0 \]

($G/H$ embeds in $\hat{G}/H$, so we define a homomorphism by $gH \mapsto g \cdot x_0$ and this extends uniquely to the completion.)

**Rk:** Having a point that is fixed by $H$ seems to be a strong condition but in fact, it happens every time $H$ is extremely amenable.

If $G/H$ is precompact and $H$ is extremely amenable, then every minimal subset of $\hat{G}/H$ is isomorphic to $M(G)$.

Indeed, if $G \times X$ is any minimal flow, then there exists $x_0 \in X$ s.t. $H \cdot x_0 = x_0$ (because $H$ is extremely amenable).

By the universality property of $\hat{G}/H$, we can find $\pi: \hat{G}/H \rightarrow X$.

\[ H \mapsto x_0 \]

If $Z \subseteq \hat{G}/H$ is minimal, then $\pi(Z) = X$ (by minimality of $X$) so $Z$ is universal.

→ Computing the universal minimal flow boils down to finding an extremely amenable subgroup $H$ of $G$ such that $G/H$ is precompact.
Ex: \textit{Glamer - Wein}

\[ G = S_\infty, \ H = \text{Aut}(\mathbb{Q}, <), \ \widehat{G}/H = L_0. \]

The only thing to check is that \( G \cdot \overline{LO} \) is minimal.

Let \( U \subseteq L_0 \) be an open set: it is given by a finite condition, say \( 1 < 3 < 4 < 5 \).

Then, given any \( x \in L_0 \), it is easy to find \( g \in G \) s.t. \( g \cdot x \in U \).

So \( M(S_\infty) = L_0 \).

And this works in general.

\* 

Main question: Is the universal minimal flow \( M(L_0) \) always of the form \( G/H \)?

[under some assumptions on \( G \)]

Remark: It is not true if \( G \) is discrete.

Reasonable assumptions:

\* \( G \) is oligomorphic

\* \( G = \text{Aut}(M) \), with \( M \) homogeneous in a finite relational language

\* \( G = \text{Aut}(M) \), with \( M \) defined by finitely many forbidden configurations.

The answer is yes in all known examples. But then, this is also the only method we know how to calculate universal minimal flows.

Still, \( G/H \) seems to be easy to classify.

For the answer to be no, we'd have to find a structure \( M \) in which Ramsey fails so badly that it still fails after expanding the language...
Properties of $\hat{G}/H$:
1) $\hat{G}/H$ is metrizable (it has a metrizable uniformity).

**Rk:** This is rare for universal minimal flows; for instance, this is not true for locally compact groups (Veech).

2) $\hat{G}/H$ contains a dense $G\delta$ orbit:
- $\hat{G}/H$ embeds homeomorphically into $\hat{G}/H$ and is a distinguished orbit.

Now the main question can split into the following two.

**Questions:**
1) When is $M(G)$ metrizable?
2) When does any metrizable minimal flow of $G$ have a $G\delta$ orbit?

**Rk:** If $M(G)$ has a $G\delta$ orbit, then so does every minimal flow of $G$.

We can attack these two questions separately. And there is no combinatorics involved.

**Rk:** For Roelcke-precompact groups, there is no known counter-example to question 1.

**Rephrasings of question 2:**
- Does the ordering property (in the sense of Nešetřil; this is a weakening of the Ramsey property) imply the weak amalgamation property (in the sense of Kechris-Koszmider) [under some assumptions of homogeneity]?

Let $M$ be homogeneous and $\omega$-categorical, $L^*$ some expansion of the language and $T^* = \text{Th}(M) \cup L^*$, where $L^*$ is a universal theory in $L^*$: Does there exist a model of $T^*$ that is $\omega$-categorical?
Theorem: Melleray - Nguyen van Thé - Tsankov

Let $G$ be a Polish group. Assume that $M(G)$ is metrizable and has a $G^d$ orbit $G \cdot x_0$.

Let $H = G_{x_0}$ be the stabilizer of $x_0$.

(Rk: $M(G)$ has only one $G^d$ orbit so $H$ is distinguished.)

Then $G/H$ is precompact, $M(G) = \hat{G}/H$ and $H$ is extremely amenable.

This theorem tells you that if you can answer these two questions for $G = \text{Aut}(M)$, you get a precompact Ramsey expansion of $M$.

Proof: Step 1: $M(G) = \hat{G}/H$.

Set $Y = G/H$.

Let $S(Y)$ be the maximal ideal space of $\text{UCB}(Y)$.

(If $Y$ is precompact, $S(Y)$ is the compactification.)

We define $j: Y \to M(G)$ (this is well-defined because $x_0$ is fixed by $H$).

$gH \mapsto g \cdot x_0$

$j$ is a homeomorphism on its image: since $M(G)$ is metrizable and $G \cdot x_0$ is $G^d$, we can apply Effros' theorem.

Now we use the universality properties of $S(Y)$ and $M(G)$ to get homomorphisms $\Psi: S(Y) \to M(G)$ such that $\Psi_0 \circ j = j$ and $\Psi: M(G) \to S(Y)$.

This is an equivalence.
Our goal is to show that \( U \) and \( \Psi \) are isomorphisms.

The first thing we do is to compose them:

\[ \Psi \circ U : M(G) \rightarrow M(G), \]

by coalescence, is an automorphism.

So by precomposing \( U \) with an automorphism (it was somehow free, it just witnesses the universality of \( M(G) \)), we can suppose that \( U \circ U \) is the identity.

Now set \( y_0 = H \in Y \) and \( z_0 = U(x_0) \).

We know that \( j(y_0) = x_0 \).

\[ S(Y) \]

\[ \text{S(Y)} \]

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Since \( Y \) is dense in \( S(Y) \) and the action of \( G \) on \( Y \) is transitive, there exists a set \( \{ \tau \} \subseteq G \) such that \( \tau \cdot j(y_0) \) converges to \( z_0 \).

We apply \( U \circ U \) to get \( \tau \cdot z_0 \rightarrow z_0 \)

\[ U \circ U(\tau \cdot j(y_0)) = \tau \cdot U \circ U(j(y_0)) = \tau \cdot z_0 \]

\[ = j(y_0) \]

\[ = z_0 \]

and \( U \circ U(z_0) = U \circ U \circ U(x_0) = z_0 \).

Then \( j^{-1} \circ U \) to get \( \tau \cdot y_0 \rightarrow y_0 \)

\[ \text{it is a homeomorphism} \]

\[ j^{-1} \circ U(\tau \cdot z_0) = \tau \cdot j^{-1} \circ U(z_0) = \tau \cdot j^{-1}(x_0) = \tau \cdot y_0 \]

then we apply \( i \) and get \( \tau \cdot j(y_0) \rightarrow j(y_0) \).
Yo we obtain that $i(y_0) = y_0$.

This means that $\Phi$ and $\Psi$ are indeed isomorphisms.

So $M(G) = S(Y) = \hat{G}/H$.

- **Step 2**: $H$ is extremely amenable.

**Lemma**: Assume that $M(G) = \hat{G}/H$.

Then $\text{Aut}(M(G)) = N(H)/H$ and the normalizer of $H$.

Every $H$-fixed point in $M(G)$ is of the form $\sigma(y_0)$, where $\sigma \in \text{Aut}(M(G))$ and $y_0 = H \in G/H$.

**Proof (of the lemma)**: Let $\sigma \in \text{Aut}(M(G))$.

Then $\sigma(G \cdot y_0) = G \cdot y_0$ because $\sigma$ has to send $G$-orbits to $G$-orbits and $G \cdot y_0$ is the only one.

Then $\sigma(y_0) = n \cdot y_0$ for some $n \in G$.

And in order for $\sigma(gH) = gnH$ to be well-defined, we must have $n \in N(H)$.

It is now easy to see that $\text{Aut}(M(G))$ and $N(H)/H$ are isomorphic. (Since they are Polish and isomorphic via a Borel map, they are homeomorphic as well.)

If $y_1$ is any other $H$-fixed point, then by universality of $\hat{G}/H$, there is a $G$-map $\pi: \hat{G}/H \rightarrow \hat{G}/H$.

$y_0 \mapsto y_1$

By coalescence ($M(G) = \hat{G}/H$), $\pi$ must be an automorphism. □(lemma)
Ex: \( G = S_\infty \)
\[ H = \text{Aut}(\mathbb{Q}, <) \]
Then \( N(H) = \text{Aut}(\mathbb{Q}, \text{betweenness}) \)
and \( N(H)/H = \text{flip} = \mathbb{Z}/2\mathbb{Z} \).

Back to the proof of step 2: we restrict to the case when \( G \leq S_\infty \).
It then suffices to check that for every open subgroup \( V \leq H \) and every \( z_0 \in \mathbb{Z}^{H/V} \), \( H \cdot z_0 \) contains a fixed point.

Rk: That corresponds to the Ramsey theorem.
Let \( V \) be an open subgroup of \( G \) such that \( V \cap H = V' \).
Note that \( H/V \) embeds into \( G/V \).
Let \( t_0 \in \mathbb{Z}^{G/V} \) be such that \( t_0 \bar{H}V'z_0 = \bar{z}_0 \).
Consider the diagonal action \( G \times G \cdot t_0 \times G/H \).

By universality of \( M(G) \), there exists a \( G \)-map
\[
\pi: M(G) = G/H \rightarrow \mathbb{Z}^{G \cdot t_0 \times G/H}
\]
\[
y_0 \rightarrow (t_1, y_1)
\]
Then \( (t_1, y_1) \) is a fixed point for \( H \) (because \( y_0 \) was) so in particular, \( y_1 \in G/H \) is a fixed point for \( H \).

By precomposing \( \pi \) with an automorphism of \( M(G) \),
we can suppose that \( y_1 = y_0 \) (this is what the lemma gives us).

Our goal is now to prove that \( t_1 \in H \cdot t_0 \).
This will suffice: we will get the result by projection (restriction) on \( \mathbb{Z}^{H/V} \).
Let \((g_n)\) be a sequence of elements of \(G\) such that \(g_n \cdot (x_0, y_0)\) converges to \((x_1, y_0)\).

We have in particular that \(g_n \cdot y_0 \to y_0\).

Let \(F \subseteq G\) be finite.

We want to approximate \(x_1\) on a finite set of cosets by \(x_0\) on an \(H\)-translate of the finite set.

We want to find \(h \in H\) s.t. \(t_1(\pi_{F,V}) = t_0(h^{-1} \pi_{F,V})\).

Let \(U = \bigcap_{f \in F} V_f\). It is an open set.

Then there exists \(m\) s.t. \(g_m H \subseteq UH\) (so \(g_m = uh\), for \(u \in U, h \in H\)) and \(t_1(\pi_{F,V}) = t_0(h^{-1} \pi_{F,V})\).

Take \(f \in F\).

We now calculate:

\[
t_1(fV) = t_0(g_m^{-1} fV) = t_0(h^{-1} u^{-1} fV)
\]

\(u^{-1}\) stabilizes \(fV\), by the choice of \(U\).

\[
= t_0(h^{-1} fV),
\]

which completes the proof.

\(\square\)

**Ex:**

\(G = \text{Aut}(\mathbb{R})\)

\(H = \text{Aut}(\mathbb{R}, <)\)

Then \(G/H\) is the set of all ordered graphs whose underlying set is the random graph.

* We will now present a simpler proof of the following theorem of Angel, Kechris and Lyons.
Let $M$ be any of the random graph, the $K_m$-free random graph, let $G = \text{Aut}(M)$. The action of $G$ on $\text{LO}(M)$ (which happens to be the universal minimal flow for all of these groups) is uniquely ergodic, i.e. there exists a unique invariant measure.

The «standard» measure on $\text{LO}(M)$ is obtained in a very simple way: 

$$
\pi \downarrow
$$

$$
(\text{LO}(M), \mu)
$$

where $\pi$ is defined by $a <_{\pi(x)} b \iff x(a) < x(b)$.

($a, b \in M$)

This measure $\mu$ is invariant under $\text{Sym}(M)$: for distinct $a_1, \ldots, a_n$, $\mu(a_1 < \ldots < a_n) = \frac{1}{n!}$ so $\mu$ is some kind of a uniform measure.

And in fact, $\mu$ is the only measure which is invariant under $\text{Sym}(M)$ (this was noticed by Glamer and Weiss when they computed the universal minimal flow of $\text{Sym}(M)$).

What the theorem says is that if we take a smaller group, this standard measure is still the only invariant measure.

In the proof, we are going to use the following theorem from the workshop. This is a 1-dimensional version of de Finetti's theorem.
Th: Let $M$ be an $\omega$-categorical structure which admits weak elimination of imaginaries and such that for all finite $A \subseteq M$, $acl(A) = A$. Let $G = \text{Aut}(M)$ and assume that the action $G \times M$ is transitive.

It applies to our situation.

Then, if $\mu$ is any $G$-invariant ergodic measure on $[0,1]^M$, then $\mu = \nu^M$ for some measure $\nu$ on $[0,1]$.

In particular, it is $\text{Sym}(M)$-invariant.

Rk: The conclusion is very similar to what we want to obtain, but the assumption is a little different: $\text{LO}(M) \leq 2^{M^2}$ and $M^2$ does not admit weak elimination of imaginaries.

Proof (AKL): Let $\mu$ ergodic and $G$-invariant be given.

Fix a «finite exhaustion» $F_1 \subseteq F_2 \subseteq \ldots$ of $M$: finite sets $F_n$ s.t. $\bigcup F_n = M$.

For $a \in M$, let $D(a) = \{ b \in M \mid b \neq a \text{ and } b \not\sim_a \}$. 

Key lemma [a law of large numbers]:

Let $a \in M$ and let $A \subseteq D(a)$ be infinite. (we will need this to get transitivity)

Then for $\mu$-almost every $\varepsilon$,

$$\lim_{n \to \infty} \frac{\# \{ b \in F_n \cap A \mid b \not\sim_{\varepsilon} a \}}{\# F_n \cap A}$$

exists and is independent of $A$ (for a predefined countable collection of subsets $A$ of $D(a)$).
Proof (of the key lemma): Consider the following $G_a$-map:

\[
\begin{array}{c}
(G_a \cap \text{LO}(M), \mu) \\
\downarrow \pi \\
(G_a \cap \text{LO}(D(a)), \tilde{\pi}_x \mu).
\end{array}
\]

We apply the theorem to the structure $D(a)$, whose automorphism group is $G_a$.

$\tilde{\pi}_x \mu$ is not necessarily ergodic but the ergodic decomposition theorem says that it splits as an integral of ergodic measures.

So $\tilde{\pi}_x \mu = \int \overline{\text{Ber}(p)} D(a) \, d\nu(p)$ for some measure $\nu$ on $[0, 1]$, where $\overline{\text{Ber}(p)}$ is the Bernoulli measure of parameter $p$.

We are now in good shape to apply the law of large numbers:

set $B = \{ x \in \text{LO}(M) |$ the limit exists and is the same $\}$ for all $A \subseteq D(a)$ in consideration

Then $K_p(\pi(B)) = 1$ for all $p$ (the limit will be $p$), which implies that $\tilde{\pi}_x \mu(\pi(B)) = 1$ and thus $\mu(B) = 1$.

Define $\eta_a(x) = \lim_{n \to \infty} \frac{\#\{b \in F_n \cap D(a) | b <_x a\}}{\# F_n \cap D(a)}$.

This is a random variable.

Now is a good time to apply the theorem a second time.
Consider the map \((\mathbb{L}^0(M), \mu)\) defined by
\[
\pi(x)(a) = \eta_a(x).
\]
Here, the measure \(\pi \times \mu\) is ergodic.
So the theorem gives that the \(\eta_a\)'s are independent identically distributed.

1) Almost surely, \(a \prec x b \Rightarrow \eta_a(x) \leq \eta_b(x)\).
   Indeed, let \(x\) be such that \(a \prec x b\).
   We have
   \[
   \eta_a(x) = \lim_{n \to \infty} \frac{\# \{c \in F_n \cap D(a) \cap D(b) \mid c \prec x a\}}{\# F_n \cap D(a) \cap D(b)}
   \]
   almost surely, by the key lemma (\(D(a) \cap D(b)\) is the set of all points that are connected neither to \(a\) nor to \(b\), and this is infinite in \(M\)).
   Now if \(c \prec x a\), then \(c \prec x b\) (because \(a \prec x b\))
   so \(\eta_a(x) \leq \lim_{n \to \infty} \frac{\# \{c \in F_n \cap D(a) \cap D(b) \mid c \prec x b\}}{\# F_n \cap D(a) \cap D(b)} = \eta_b(x)\).

2) For all \(a \neq b\), almost surely \(\eta_a \neq \eta_b\).
   Suppose not.
   Then because of the independence of the \(\eta_a\)'s,
   there has to be an atom: there exists \(p \in [0,1]\)
   s.t. \(P(\eta_a = p) > 0\).
   So this event will happen almost surely for infinitely many \(a\)'s.
   In particular, for fixed \(a, b, c\) with \(a \neq b\),
   \[
   \begin{cases}
   a \prec x b \prec x c \\
   \eta_a = \eta_b = \eta_c = p
   \end{cases}
   \]
   has positive probability.
   (If not, by invariance, it would happen nowhere.)
Let $q = P(a <_x b <_x c \mid \eta_a = \eta_c = \uparrow$ and $a <_x c$).

This conditional probability is positive.

Fix $a$ and $c$.

If $b$ varies in $D(a) \cap D(c)$, $q$ does not change
($q$ depends only on the type of $a$ and $c$).

We compute the expected value:

\[
E\left(\frac{\# \{ b \in F_n \cap D(a) \cap D(c) \mid a <_x b <_x c \}}{\# F_n \cap D(a) \cap D(c)} \mid \eta_a = \eta_c = \uparrow \quad \text{and} \quad a <_x c\right) = q
\]

For each $b$, the probability that this happens is $q$.

But we can also take the limit:

\[
\lim_{n \to \infty} E\left( \ldots \right) = q
\]

and $E\left( \lim_{n \to \infty} \ldots \right) = E(\eta_c - \eta_a \mid \eta_a = \eta_c = \uparrow) = 0$

but they should be equal, a contradiction.

**Conclusion:** Almost surely, if $a <_x b$ then $\eta_a(x) < \eta_b(x)$.

Since $<_x$ is a linear ordering, it goes both ways: almost surely,

\[a <_x b \iff \eta_a(x) < \eta_b(x).\]

Then $LO(M)$ is 1-to-1

\[\downarrow\]

\[[0,1]^M\]

and the measure on $[0,1]^M$ is Sym$(M)$-invariant

so $\mu$ is the standard measure. \hfill \Box

**Rk:** A version of this proof works for the generic partial order.