The universal minimal flow of the group of automorphisms of $\mathcal{P}(\omega_1)/\mathrm{fin}$

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Katowice problem

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RATHER NOT: very trivial automorphism on ω_1 transfers to nontrivial automorphism of $\mathcal{P}(\omega)/\text{fin}$ (K.P. Hart)

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Universal minimal flows

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Theorem (folklore)

The universal minimal flow exists and it is unique up to an isomorphism.

Automorphism groups and the space of linear orderings

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for $A \in \text{Age}(\mathcal{A})$.

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$\mathcal{K}_<$ an order expansion of Age($\mathcal{A}$) is order forgetful if

$$(A,<),(B,\prec) \in \mathcal{K} \text{ and } A \cong B \text{ imply } (A,<) \cong (B,\prec)$$

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(HP) Hereditary property: if A is a finitely-generated substructure of B and $B \in \mathcal{F}$, then also $A \in \mathcal{F}$.

(JEP) Joint embedding property: if $A, B \in \mathcal{F}$ then there exists a $C \in \mathcal{F}$ in which both A and B embed.

(AP) Amalgamation property: if $A, B, C \in \mathcal{F}$ and $i : A \longrightarrow B$ and $j : A \longrightarrow C$ are embeddings, then there exist $D \in \mathcal{F}$ and embeddings $k : B \longrightarrow D$ and $l : C \longrightarrow D$ such that $k \circ i = l \circ j$.

Ramsey classes

A class $\mathcal{K}$ of finite structures is a Ramsey class if for every $A \leq B \in \mathcal{K}$ and $k \geq 2$ a natural number there exists $C \in \mathcal{K}$ such that

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Example

- Finite linear orders (Ramsey Theorem);
- finite linearly ordered (K_n-free) graphs;
- finite linearly ordered hypergraphs;
- finite Boolean algebras (Dual Ramsey Theorem);
- finite vector spaces over a given finite field.

Theorem (DB; KPT for $\mathcal{A}$ countable)

- $G = \mathrm{Aut}(\mathcal{A})$ *with pointwise convergence topology*
- $\mathcal{A}$ *- an order-forgetful* ω*-homogeneous structure*
- $\mathrm{Age}(\mathcal{A})$ *- consists of finite structures*
- $\mathcal{K}_<$ *- linear Fraïssé expansion of* $\mathrm{Age}(\mathcal{A})$
- $\mathcal{K}_<$ *- Ramsey class*
- $\mathcal{K}_<$ *- consists of rigid structures.*

Then the universal minimal flow of G *is the space of linear orderings on* $\mathcal{A}$ *whose restrictions to finite substructures of* $\mathcal{A}$ *belong to* $\mathcal{K}_<$*.*

Universal minimal flows of automorphism groups

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Remark

The class of finite Boolean algebras is a Ramsey class but does not consit of rigid elements.

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Theorem (Glasner, Gutman (2012))

The universal minimal flow of $\mathrm{Aut}(\mathcal{P}(\omega)/\mathrm{fin})$ *(and therefore of* $\mathrm{Aut}(\mathcal{P}(\omega_1)/\mathrm{fin})$*) is the space of maximal chains of closed subsets of* $\omega^* = \beta\omega \setminus \omega$.

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Theorem

The universal minimal flow of $\mathrm{Aut}(\mathcal{P}(\omega)/\mathrm{fin})$ *is the space of linear orderings on* $\mathcal{P}(\omega)/\mathrm{fin}$ *that are natural when restricted to a finite subalgebra.*

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$M(\mathrm{Aut}(\mathcal{P}(\omega_1)/\mathrm{fin}))$ *is the space of linear orderings on* $(\mathcal{P}(\omega_1)/\mathrm{fin}, \mathrm{I})$ *that are I-natural when restricted to any finite substructure.*

Ramsey classes of finite Boolean algebras with ideals

- **Fraïssé Ramsey class**
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THANK YOU FOR YOUR ATTENTION!