Continuous first-order model theory for metric structures
Lecture 3 (of 3)

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The Urysohn space of diameter ≤ 1 (continued from Lecture 2, for a few slides)

Notation from the last lecture: $\mathbb{U}$ is the unique separable model of the model companion T^* (of the theory of metric spaces of diameter ≤ 1) and $\mathbb{M}$ is an elementary extension of $\mathbb{U}$ that is a monster model of T^*.

It is often important to consider types over arbitrary sets of parameters. So take a small set $A \subseteq \mathbb{M}$ and consider $\mathrm{S}_n(A)$, which we view as the space of n-types for the theory of $(\mathbb{M}, a)_{a \in A}$.

We may assume A is a closed set. Indeed the restriction map $\mathrm{S}_n(\overline{A}) \to \mathrm{S}_n(A)$ is a homeomorphism and an isometry. Or we may replace A by any dense subset (e.g., if we want to keep the size of the signature of $(\mathbb{M}, a)_{a \in A}$ smaller).

As noted before, QE for the theory of $\mathbb{U}$ allows us to identify $\mathsf{S}_1(A)$ with the set $K_{\leq 1}(A)$ of Katětov functions with values in $[0,1]$. The correspondence is a homeomorphism (for the logic topology and the topology of pointwise convergence) and an isometry (for the induced metric and the supremum metric). (This last fact depends on the optimal amalgamation lemma for metric spaces.)

More generally, using QE for the theory of $\mathbb{U}$, we may identify $\mathsf{S}_n(A)$ with a closed subset of the product $\mathsf{S}_1(A)^n \times \mathcal{M}$, where $\mathcal{M}$ is the closed subset of $[0,1]^{n^2}$ consisting of arbitrary distance matrices of n-tuples from a metric space of diameter ≤ 1. This identification relates the metric on $\mathsf{S}_n(A)$ with the coordinate maximum metric obtained from the given metrics on the $n+1$ factors in the product.

Corollary

Let A be any set of parameters in $\mathbb{M}$. The following are equivalent
(1) $S_n(A)$ is metrically compact for all $n \geq 1$.
(2) $S_1(A)$ is metrically compact.
(3) A is relatively compact.

Therefore, $\mathbb{U}$ is homogeneous for isometric copies of A if and only if A is relatively compact. (Also noticed by Melleray and Ben Ami, perhaps others, and for the unbounded Urysohn space, not just the bounded one.)

Proposition

If A is not relatively compact, then $S_1(A)$ has metric density 2^ω.

When A is not relatively compact, it becomes interesting to determine what the isolated types are and when/if they are dense. Toward that end, we have the following characterizations of isolated 1-types over A.

Proposition

For any 1-type p over A, the following are equivalent.
(1) p is isolated in $S_1(A) = K_{\leq 1}(A)$.
(2) For every $\epsilon > 0$ there is a finite $F \subseteq A$ such that for every $q \in S_1(A)$ we have

$$p|F = q|F \quad \Rightarrow \quad d(p, q) < \epsilon$$

(3) For every $\epsilon > 0$ there is a finite $F \subseteq A$ such that for every $a \in A$ we have

$$\max_{u \in F} |p(u) - d(a, u)| + \epsilon \geq \min_{u \in F}(p(u) + d(a, u))$$

Gurarij's universal homogeneous Banach space

Recall the signature for unit balls of Banach spaces is $\mathcal{L} = \{0, c_{r,s}, \| \ \|\}$ where r, s range over rational scalars such that $|r| + |s| \leq 1$. This is a countable signature.

- 0 is a constant symbol.
- $c_{r,s}$ are binary function symbols.
  - Interpret $c_{r,s}(x, y)$ as $rx + sy$.
- $\| \ \|$ is a unary predicate symbol.
- the functions and predicate are 1-Lipschitz in each variable.

Notation

We let T_b denote a theory axiomatizing the class of all (unit balls of) Banach spaces.

Definition (introduced by V I Gurarij in the mid 1960s)

A Banach space X will be said to have the Gurarij property if for every $S\colon E \to X$, every $F \supseteq E$, and every $\lambda > 1$, there is $T\colon F \to X$ such that T extends S, $\|T\| \leq \lambda\|S\|$ and $\|T^{-1}\| \leq \lambda\|S^{-1}\|$.
(Here E, F are finite dimensional and S, T are injective linear.)

Definition

A Gurarij space is a separable Banach space with the Gurarij property.

A few facts are reasonably obvious:

- A Banach space X has the Gurarij property if and only if the unit ball of X is e.c. as a model of T_b.
- Any two Gurarij spaces are λ-isomorphic for all $\lambda > 1$.
- Every Banach space X embeds in one with the Gurarij property; if X is separable it embeds in some Gurarij space.

Tools of Banach space geometry in the late 1960s were adequate to prove:

- If X is a Gurarij space, then the dual space X^* is linearly isometric to $L_1(\mu)$ for some measure μ and X contains subspaces linearly isometric to $\ell_\infty(n)$ for every $n \in \mathbb{N}$.
- No Gurarij space can satisfy the condition in the definition with $\lambda = 1$.

Notation

Let $\mathcal{EC}(T_b)$ be the class of e.c. models of T_b. As noted above, $\mathcal{EC}(T_b)$ is also the class of (unit balls of) Banach spaces with the Gurarij property.

Proposition

The class $\mathcal{EC}(T_b)$ is axiomatizable. Therefore, T_b has a model companion, which we will denote T_b^*; the models of T_b^* are exactly the members of $\mathcal{EC}(T_b)$.

Corollary

The model companion T_b^* is complete and has quantifier elimination.

QE for T_b^* implies that its types have a natural description.

Consider $\mathcal{M} \vDash T_b^*$ and $a \in M^n$. We may identify $p = \mathrm{tp}_{\mathcal{M}}(a)$ with the function $t_p \colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$t_p(r) = \| \sum r_i a_i \|.$$

Indeed, t_p is determined by its restriction to the set

$$\{ r \in \mathbb{R}^n \mid \sum |r_i| = 1 \}.$$

The key to writing axioms for the class $\mathcal{EC}(T_b)$ as well as to verifying more subtle properties of T_b^* is the following "optimal" amalgamation property of Banach spaces:

Lemma

Let X, Y be Banach spaces and I an index set; consider families $a = (a_i)_{i\in I} \in X^I, b = (b_i)_{i\in I} \in Y^I,$ and $\varepsilon = (\varepsilon_i)_{i\in I} \in (\mathbb{R}^{\geq 0})^I$. Also let $\mathbb{R}^{(I)}$ denote the set of all families in $\mathbb{R}^I$ in which only finitely many coordinates are nonzero.
The following conditions are equivalent.

(1) There is a seminorm $\|\cdot\|$ on $X \oplus Y$ that agrees with the given norms on X and Y and satisfies $\|a_i - b_i\| \leq \varepsilon_i$ for all $i \in I$.

(2) For all $r \in \mathbb{R}^{(I)}$ one has $\big|\|\sum r_i a_i\| - \|\sum r_i b_i\|\big| \leq \sum |r_i|\varepsilon_i$.

Proof: ($\Rightarrow$) From the triangle inequality for the norm.
($\Leftarrow$) The desired seminorm on $X \oplus Y$ may be defined by

$$\|x + y\| = \inf\{\|x - \sum_i r_i a_i\| + \|y + \sum_i r_i b_i\| + \sum_i |r_i|\epsilon_i\}$$

where the inf is taken over $r = (r_i) \in \mathbb{R}^{(I)}$.

The preceding Lemma gives a precise formula for the induced metric on $S_n(T_b^*)$:

$$d(\mathrm{tp}(a), \mathrm{tp}(b)) = \sup\{\big|\|\sum_{i=1}^n r_i a_i\| - \|\sum_{i=1}^n r_i b_i\|\big| \mid \sum_{i=1}^n |r_i| = 1\}$$

Corollary

For each $n \in \mathbb{N}$, the space $S_n(T_b^*)$ is compact under the induced metric. Therefore T_b^* is separably categorical. Hence there is exactly one separable Banach space with the Gurarij property, up to isometry. In other words, the Gurarij space is unique.

Uniqueness of the Gurarij space was first proved in the mid 1970s by W Lusky. A more elementary proof was given recently by Kubiś and Solecki.

Notation

Let $\mathbb{G}$ denote the Gurarij space and $\mathbb{B}$ its unit ball. Consider a tuple $a \in \mathbb{B}^m$ and let

- $T_b^*(a) = \mathrm{Th}(\mathbb{B}, a_1, \ldots, a_m)$, and
- $\mathrm{S}_n(a) = \mathrm{S}_n(T_b^*(a))$

Note that $T_b^*(a)$ and $\mathrm{S}_n(a)$ only depend on the quantifier-free type of a, because T_b^* admits quantifier elimination.

Suppose $p \in \mathrm{S}_n(a)$ is realized by b; then p can be identified with the function

$$t_p(r,s) = \|\sum r_i a_i + \sum s_j b_j\|$$

where r, s come from $\mathbb{R}$ and $\sum |s_j| = 1$.

Moreover, if $p, q \in \mathrm{S}_n(a)$ are realized by b, c, then

$$d(p,q) = \sup\{|t_p(r,s) - t_q(r,s)| \mid r, s \text{ come from } \mathbb{R} \text{ and } \sum |s_j| = 1\}$$

Note: when E is any nonzero Banach space, with $E = \mathrm{span}(a)$, one can use the above formula for the metric on $\mathrm{S}_1(a)$ to show by a direct construction that there exist infinitely many types in $\mathrm{S}_1(a)$ that are uniformly separated by distance 1. Therefore theories such as $T_b^*(a)$ are not separably categorical as long as a includes at least one nonzero vector.

Let E be a finite dimensional subspace of $\mathbb{G}$. Let $\mathcal{S}(E, \mathbb{G})$ be the set of all isometric linear embeddings of E into $\mathbb{G}$; it is a Polish space on which the automorphism group of $\mathbb{G}$ acts continuously. We are interested in the complexity of the orbit space of this action.

What follows in this section is the result of ongoing joint work with Itaï Ben Yaacov.

Polyhedral parameter spaces

Theorem

Suppose $a \in \mathbb{B}^m$ and $E = span(a)$ in $\mathbb{G}$ is polyhedral.

- *For each $n \in \mathbb{N}$, the space $\mathrm{S}_n(a)$ is separable in the induced metric.*
- *Therefore $T_b^*(a)$ has a separable atomic model, which can be taken to be of the form $(\mathbb{B}, b)$.*
- *The set of all $b \in \mathbb{B}^m$ for which $(\mathbb{B}, b)$ is an atomic model of $T_b^*(a)$ is a full orbit under the action of the automorphism group of $\mathbb{G}$.*
- *(e.g.) For nonzero singletons $b \in \mathbb{B}$, we know $(\mathbb{B}, b)$ is an atomic model iff b is smooth in $\mathbb{G}$.*

Theorem

Suppose E is a polyhedral finite dimensional Banach space, with vector basis $a = a_1, \ldots, a_n$. The set of $S \in \mathcal{S}(E, \mathbb{G})$ such that $(\mathbb{G}, S(a))$ is an atomic model is a dense G_δ set in $\mathcal{S}(E, \mathbb{G})$ and a full orbit under the action of $\mathrm{Aut}(\mathbb{G})$.

In particular, when $n = 1$ and $a \in \mathbb{G}$ satisfies $\|a\| = 1$, we have the result (indicated by Lusky at the end of his uniqueness paper) that the set of smooth points of norm 1 in $\mathbb{G}$ is a full orbit under the action of the automorphism group of $\mathbb{G}$. Mazur proved that the set of smooth points of norm 1 is a dense G_δ subset of the unit sphere in every separable Banach space.

What if E is not polyhedral?

Theorem

Suppose $a \in \mathbb{B}^m$ and the linear span of a in $\mathbb{G}$ is not polyhedral. Then $S_1(a)$ has density 2^ω (the maximum possible) in the induced metric.

Proof: J. Lindenstrauss in 1964 constructed a sequence (b_n) in $E = \text{span}(a)$ such that for any $m \neq n$ and any choice of signs, $\|b_m \pm b_n\| \leq \|b_m\| + \|b_n\| - 1$. Embed E isometrically in ℓ_∞ and for each choice of signs $\epsilon = (\epsilon_n)$ consider the family of closed balls of radius $\|b_n\| - \frac{1}{2}$ around $\epsilon_n b_n$. By hypotheses, each two of these balls have nonempty intersection. Hence there exists c_ϵ that belongs to all of them. One checks that the family of all (continuum many) types in $S_1(E)$ realized by the family (c_ϵ) is uniformly 1-separated. Hence the metric density of $S_1(a)$ is the continuum. □

- V I Gurarij, *Space of universal disposition, isotopic spaces and the Mazur problem on rotations of Banach spaces*, Siberian Math. Journal 7 (1966), 799–806 (translation from Russian).
- W Lusky, *The Gurarij spaces are unique*, Archive of Math. 27 (1976), 627–635.

Definable sets

Basic setting: T is a theory and we have a definable predicate P (of n arguments) in models of T. Recall this means we have a continuous function $\pi = \pi_P \colon S_n(T) \to [0,1]$ that gives the interpretation of P in any $M \models T$ by the equation (for each $a \in M^n$)

$$P^M(a) = \pi(\mathrm{tp}_M(a))$$

Equivalently, we have a sequence $(\varphi_n(x)) = (\varphi_{P,n}(x))$ of formulas such that (for each $M \models T$ and each $a \in M^n$)

$$P^M(a) = \lim_n \varphi_n^M(a)$$

and we require that the convergence in this limit is uniform in the tuple a and in the model M.

For such a situation, we let Z_P^M to be the zeroset of P^M, whenever $M \vDash T$. That is

$$Z_P^M = \{a \in M^n \mid P^M(a) = 0\}$$

Note that Z_P^M is always a type-definable set; indeed for a suitable sequence (ϵ_n) of reals, we have

$$Z_P^M = \bigcap_n \{a \in M^n \mid \varphi_n^M(a) \leq \epsilon_n\}$$

for every $M \vDash T$.

Moreover, for any $M \vDash T$ and $a \in M^n$ we see that $a \in Z_P^M$ if and only if $\mathrm{tp}_M(a)$ is in the closed subset of $\mathrm{S}_n(T)$ given by

$$\pi_P^{-1}(0) = \bigcap_n \{q \in \mathrm{S}_n(T) \mid \varphi_n(x) \leq \epsilon_n \text{ is an element of } q\}$$

which we will denote by $[P = 0]$.

Finally, note that if the signature of T is countable, then every closed subset of $\mathrm{S}_n(T)$ is of the form $[P = 0]$ for some definable predicate P.

Definition

We say the zerosets of P are definable in models of T if the predicate on M^n giving the distance to Z_P^M is a definable predicate in models M of T.

That is, there should be a definable predicate Q in models of T such that for every $M \models T$ and every $a \in M^n$ we have

$$Q^M(a) = \mathrm{dist}(a, Z_P^M)$$

The preceding definition applies, of course, to formulas. In particular, we do not automatically regard the zeroset of the interpretation of a formula as being a definable set, in contrast to what is done in classical model theory. The reason for this will be explained in a few slides.

Lemma

The zerosets of P are definable in models of T if and only if the function (from $S_n(T)$ to $[0,1]$) given by

$$dist(q, [P = 0])$$

is continuous.

Proof: For every $M \models T$ and $a \in M^n$ we have

$$\text{dist}(a, Z_P^M) = \text{dist}(tp_M(a), [P = 0])$$

Note: we sometimes use the phrase the closed subset $[P = 0]$ is definable in $S_n(T)$ as an equivalent way to say that the zerosets of P are definable in models of T. That is, a closed subset X of $S_n(T)$ is definable if the function $f(q) = \text{dist}(q, X)$ is continuous on $S_n(T)$.

The following result explains why this is the “right” definition of “definable set” in the model theory of metric structures:

Theorem

Let P be an n-ary definable predicate in models of T. The following are equivalent.
(1) The zerosets of P are definable in models of T,
(2) For any $(m+n)$-ary definable predicate Q in models of T, there is an m-ary definable predicate R in models of T such that for any $M \models T$ and any $a \in M^m$ we have

$$R^M(a) = \inf\{Q^M(a,b) \mid b \in M^n \text{ and } P^M(b) = 0\}$$

That is, we can “quantify” over the zeroset of P without leaving the category of definable predicates.

Proof: ($\Leftarrow$) Apply condition 2 to the formula $d(x, y)$.
($\Rightarrow$) By uniform continuity of Q^M and a real analysis argument there exists a continuous function $\alpha\colon [0,1] \to [0,1]$ with $\alpha(0) = 0$ such that

$$|Q^M(a, b) - Q^M(a, c)| \leq \alpha(d(b, c))$$

for all $a \in M^m$ and all $b, c \in M^n$; moreover, α can be taken independent of M. Now check that

$$\inf\{Q^M(a, b) \mid b \in Z_P^M\} = \inf\{Q^M(a, c) + \alpha(d(c, Z_P^M)) \mid c \in M^n\}$$

and note that the right side is a definable predicate when condition (1) holds.

Criterion 1 for definability of zerosets

Proposition

The following are equivalent for a definable predicate P in models of T.
(1) The zerosets of P are definable sets in models of T.
(2) For all $\epsilon > 0$ there exists $\delta > 0$ such that for all $M \models T$ and all $a \in M^n$ we have

$$P^M(a) < \delta \text{ implies } \exists b \in M^n (d(a,b) \leq \epsilon \text{ and } P^M(b) = 0)$$

Condition (2) says that the sets $\{a \in M^n \mid P^M(a) < \delta\}$ Hausdorff converge to the zeroset of P^M in M^n. Note that condition (1) depends only on the zerosets of P and not on P otherwise.

Proof ($\Rightarrow$) Let Q be the definable predicate with interpretations $\mathrm{dist}(a, Z_P^M)$ for all $M \models T$. Note that in every model, P^M and Q^M have the same zeroset. Using a compactness argument, we see that for every $\epsilon > 0$ there exists $\delta > 0$ so that for all $M \models T$ and all $a \in M^n$ we have

$$P^M(a) \leq \delta \text{ implies } Q^M(a) < \epsilon$$

This implies condition (2).
($\Leftarrow$) Assuming condition (2) we get a continous function $\alpha \colon [0,1] \to [0,1]$ with $\alpha(0) = 0$ and $\mathrm{dist}(a, Z_P^M) \leq \alpha(P^M(a))$ for all $M \models T$ and all $a \in M^n$. Then let Q(x) be the definable predicate given by $\inf_y(\alpha(P(y)) \dot{+} d(x,y))$ and show that its interpretation is always equal to $\mathrm{dist}(a, Z_P^M)$.

Criterion 2 for definability of zerosets

Proposition

The following are equivalent for a definable predicate P in models of T.
(1) The zerosets of P are definable sets in models of T.
(2) For any family $(M_i \mid i \in I)$ of models of T and any ultrafilter $\mathcal{U}$ on I, with N the $\mathcal{U}$-ultraproduct of the family, every member of Z_P^N is represented by a family $(a_i \mid i \in I)$ with $a_i \in Z_P^{M_i}$ for a $\mathcal{U}$-large set of I.

Example 1

In the theory of probability algebras, the set of atoms (including 0) is definable in all models. To prove this, consider these formulas

$$\chi(x) = \inf_y |\mu(x \cap y) - \mu(x \cap y^c)|$$

$$\varphi(x) = \inf_z (d(x, z) \dotplus (\mu(z) \dot{-} \chi(z)))$$

and check the following statements.

Proposition

Let M be a probability algebra with measure μ and $a \in M$, and let A^M be the set of atoms in M.
(1) If a is atomless, then $\chi^M(a) = 0$.
(2) If a is an atom, then $\chi^M(a) = \mu(a)$.
(3) If a is not atomless and u is the atom of largest measure that is contained in a, then $\chi^M(a) \leq \mu(u)$.
(4) $dist(a, A^M) = \varphi^M(a)$.

Example 2

In the theory of (unit balls of) C^*-algebras, it is useful to know that the set of orthogonal projections is a definable set, since many properties are naturally axiomatized by conditions in which one quantifies over this set (or analogous ones).

To prove this definability, note that the set of projections is the zeroset of the formula $P(x) = (\|u^* - u\| \dot{+} \|u^2 - u\|)$. We show that P verifies criterion 1. Suppose u is an element of a C^*-algebra $\mathcal{A}$ making $P(u)$ very small. We consider the selfadjoint element $v = \frac{1}{2}(u^* + u)$, which must be very close to u in this situation. Now using the spectral theorem and functional calculus in $\mathcal{A}$, we produce an orthogonal projection in $\mathcal{A}$ that is very close to v and hence also very close to u. Hence u is close to the zeroset of P.

Example 3

Let T be the theory of pointed metric spaces of diameter ≤ 1 in which the metric satisfies the ultrametric inequality:

$$d(x,y) \leq \max(d(x,z), d(z,y))$$

(in a signature with c the constant symbol naming the distinguished element) and let $r \in (0,1)$. The closed balls of radius r centered at c are not definable in models of T.

To prove this, we show that criterion 1 fails to hold. Note that the closed ball of radius r is the zeroset of the formula $P(x) = (d(c,x) \dot{-} r)$. Consider a model of T containing a sequence (a_n) with the property that $d(c,a_n) > r$ for all n, and $r = \inf_n d(c,a_n)$. Let M be an ω-saturated elementary extension of this model. In $S_1(T)$ consider $p_n = \mathrm{tp}_M(a_n)$.

Using the compactness and metrizability of the logic topology on $S_1(T)$ and taking a subsequence if necessary, we may assume (p_n) converges to some type p. Let a realize p in M. It follows from the construction that $d(c,a) = r < d(c,a_n)$ for all n. Let ϵ satisfy $0 < \epsilon < r$ and suppose δ is obtained satisfying the statement of condition (2) in criterion (1). For large enough n we have $P^M(a_n) = (d(c,a_n) \dot- r) < \delta$, so condition (2) should give us b in the zeroset of P^M such that $d(a_n,b) \leq \epsilon < r$. But then we have a triangle in M with distances

$$d(a_n,b) < r = d(c,b) < d(c,a_n),$$

which violates the ultrametric inequality.

Definable sets in separably categorical theories

Theorem

The following are equivalent for a complete theory T in a countable signature.
(1) T is separably categorical.
(2) Every zeroset of a definable predicate in T is a definable set.

Proof: ($\Rightarrow$) Let X be a closed set in $S_n(T)$. The function $\text{dist}(q, X)$ is obviously continuous for the metric topology, so it is continuous since T is separably categorical.
($\Leftarrow$) Take any $p \in S_n(T)$ and consider the closed set $X = \{p\}$. Condition (2) implies that $d(p, q)$ is a continuous function of q on $S_n(T)$, from which it follows that p is isolated. Since n and p were arbitrary, this implies T is separably categorical.