Continuous first-order model theory for metric structures
Lecture 2 (of 3)

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Some conventions

When considering a theory T and its type spaces $S_n(T)$:

- topological notions always refer to the logic topology (closed set, continuous function, etc);
- metric notions always refer to the metric induced by distances in models of T.
  (ball of radius r; uniformly continuous function, etc);
- except we refer explicitly to the metric topology when we need to.

Some notation

Consider a theory T and a model $M \models T$. For each $A \subseteq M$ we let

$$\langle A \rangle_M$$

denote the substructure of M generated by A.

The underlying set of $\langle A \rangle_M$ is the closure in M of the set of all elements $t^M(a)$ where $t(x)$ is any term from the signature of T and a is any tuple from A.

Suppose $M, N \models T$ and $a \in M^n, b \in N^n$. We write

$$a \equiv_0 b \text{ in } M, N$$

if $\varphi^M(a) = \varphi^N(b)$ for every quantifier-free formula $\varphi(x)$.

Note that this is equivalent to the existence of an isomorphism J from $\langle a \rangle_M$ onto $\langle b \rangle_N$ satisfying $J(a_i) = b_i$ for all $i = 1, \dots, n$.

Likewise we write

$$a \equiv b \text{ in } M, N$$

if $\varphi^M(a) = \varphi^N(b)$ for every formula $\varphi(x)$, which is the same as saying

$$(M, a) \equiv (N, b)$$

T admits quantifier elimination if

- for every formula $\varphi(x)$ and every $\epsilon > 0$ there is a quantifier-free formula such that the following condition holds in all models of T:

$$\sup_x \left|\varphi(x) - \psi(x)\right| \leq \epsilon$$

Here $x = x_1, \ldots, x_n$ is a finite tuple and $\sup_x$ means $\sup_{x_1} \cdots \sup_{x_n}$.

Criterion 1 for QE: types = qf types

T admits QE if and only if

- for all $n \geq 1$ and all $p \in \mathrm{S}_n(T)$, the type p is determined by the quantifier free formulas it contains.

Proof: Consider the restriction map $\pi\colon \mathrm{S}_n(T) \to \mathrm{S}_n^{qf}(T)$. This map is always continuous and surjective. QE is equivalent to its being a topological homeomorphism. Since both spaces are compact, for QE it suffices that the map is injective.

Criterion 2 for QE: back-and-forth property

T admits QE if and only if

- for every ω-saturated models M, N of T and every $a \in M^n, b \in M, a' \in N^n$, if $a \equiv_0 a'$ in M, N, then there exists $b' \in N$ such that $(a, b) \equiv_0 (a', b')$ in M, N.

Proof: $(\Leftarrow)$ Use the condition to show that whenever $a \equiv_0 a'$ in M, N, then a and a' are given the same values by formulas of the form $\inf_y \varphi(x, y)$ in which φ is quantifier-free. Use an argument like that for Criterion 1 to show that every formula of this kind is approximated uniformly by quantifier-free formulas. Then use induction on syntactic complexity to show the same is true for all formulas.

Criterion 3 for QE: extension of embeddings

T admits QE if and only if

- for every $M, N \models T$, every embedding of a substructure of M into N can be extended to an embedding of M into an elementary extension of N.

Moreover, if the signature of T has $\leq \kappa$ many symbols, then it suffices to consider M of density character $\leq \kappa$ and to consider a fixed elementary extension of N that is κ-saturated.

Proof: ($\Rightarrow$) Show T verifies Criterion 2. So consider ω-saturated models M, N of T and tuples $a \in M^n$, $a' \in N^n$ and an element $b \in M$. Assume $a \equiv_0 a'$ in M, N. This gives a canonical isomorphism J of $\langle a \rangle_M \subseteq M$ onto $\langle b \rangle_N \subseteq N$ that maps a exactly to b. The condition above gives an extension of J to an embedding (which we still call J) of M into an elementary extension N' of N. This ensures $(a, b) \equiv_0 (a', J(b))$ in M, N'. Since N is ω-saturated, we can find $b' \in N$ such that $(a', J(b)) \equiv_0 (a', b')$ in N, N'. Therefore $(a, b) \equiv_0 (a', b')$ in M, N as desired.

Some examples of theories with QE:

- APr = theory of atomless probability algebras.
- H_∞ = theory of infinite dimensional Hilbert spaces.
- ALpL = theory of L_p Banach lattices of atomless measure spaces, $1 \leq p < \infty$ fixed.

(Some details given on the blackboard; the first two examples are easy using Criterion 1. The third is more complicated. It is treated using Criterion 3 as Example 13.18 in the paper *Ultraproducts in analysis* by WH and José Iovino. See the references at the end of these slides.)

$M \models T$ is an existentially closed (e.c.) model of T if

- for every $M \subseteq N \models T$, every quantifier-free formula $\varphi(x, y)$, and every $a \in M^m$, $\quad \big(\inf_y \varphi(a, y)\big)^M = \big(\inf_y \varphi(a, y)\big)^N$.

Here $y = y_1, \ldots, y_n$ is a finite tuple and $\inf_y$ means $\inf_{y_1} \cdots \inf_{y_n}$.

The condition above is equivalent to requiring the implication

$$\big(\inf_y \varphi(a, y)\big)^N = 0 \text{ implies } \big(\inf_y \varphi(a, y)\big)^M = 0$$

for all quantifier-free φ.

- T is model complete if every embedding between models of T is an elementary embedding.
- (Note: quantifier elimination $\implies$ model complete.)

- T^* is a model companion of T if they have the same signature, T^* is model complete, and T, T^* have the same substructures of models.

- T is inductive if its class of models is closed under unions of arbitrary chains.

Theorem

Suppose T is an inductive theory and $M \models T$. Then there exists an e.c. model N of T such that $M \subseteq N$.

Theorem

Let T be an inductive theory and let T^* be any theory with the same signature as T.
(a) T^* is a model companion of T if and only if the models of T^* are exactly the e.c. models of T.
(b) In particular, T has a model companion if and only if the class of e.c. models of T is axiomatizable.

Theorem

Suppose T^* is a model companion of T. If models of T have the amalgamation property over substructures, then T^* admits quantifier elimination.

Throughout this subsection on separable models, we take T to be a complete theory with a countable signature.

isolated types, atomic models

- A type $p \in S_n(T)$ is isolated (i.e., principal) if it has the same filter of neighborhoods in the logic topology as in the metric topology.

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- A type $p \in \mathrm{S}_n(T)$ is isolated (i.e., principal) if it has the same filter of neighborhoods in the logic topology as in the metric topology.
- A structure M is atomic if every type realized in M is isolated (w.r.t. $T = \mathrm{Th}(M)$).

Theorem (part of the Omitting Types Theorem)

Let $p \in S_n(T)$. The following are equivalent:
(1) p is isolated.
(2) p is realized in every model of T.

Theorem (Atomic models)

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(b) T has at most one separable atomic model.
(c) If $\mathrm{S}_n(T)$ is separable in the metric topology, for all $n \in \mathbb{N}$, then T has a separable atomic model.

- M is ω-near-homogeneous if for any $a, b \in M^n$ realizing the same n-type in M, and any $\epsilon > 0$, there is an automorphism σ of M such that $d(\sigma(a), b) < \epsilon$.

Theorem

If M is a separable atomic model of T, then M is ω-near-homogeneous. In particular, this is true of the unique separable model of a separably categorical theory.

Caution: in discrete structures (classical model theory), the countable atomic model is always ω-homogeneous. In the setting of metric structures, however, ω-near-homogeneity is often the most one has. We give an example in a few slides.

Theorem (Separable categoricity)

The following are equivalent:
(1) T has exactly one separable model.
(2) Every separable model of T is atomic.
(3) Every type in $S_n(T)$ is isolated, for all $n \in \mathbb{N}$.
(4) The metric topology is compact on $\mathrm{S}_n(T)$, for all $n \in \mathbb{N}$.
(5) The logic topology agrees with the metric topology on $\mathrm{S}_n(T)$, for all $n \in \mathbb{N}$.

Corollary

Suppose T is separably categorical and T' is the restriction of T to a smaller signature. Then T' is also separably categorical.

Note: this is true in particular if T' is the theory of all metric spaces that arise from models of T. *A priori* T' might have separable models that do not come from any model of T.

Proof: for each n, consider the restriction map from $\mathrm{S}_n(T)$ onto $\mathrm{S}_n(T')$; it is continuous, contractive, and surjective. Hence it preserves compactness with respect to the metric topologies.

An example of non-homogeneous separably categorical model

Let T_0 be the theory of all unit ball structures obtained from the L_1 Banach lattice of an atomless measure space. Note that T_0 is separably categorical, with unique separable model the unit ball B of the L_1 Banach lattice based on $[0, 1]$ with Lebesgue measure. As discussed above, T_0 admits QE.

Let T_1 be the theory of all structures (M, f) where M is is a model of T_0 and $f \in M$ satisfies $f \geq 0$ and $\|f\|_1 = 1$. T_1 is a complete theory and admits QE itself. Its separable models are of the form (B, f).

There are two isomorphism classes of separable models of T_1. In one of them, the support of f has full measure as a subset of $[0,1]$, and in the other, f is supported on a set $\subseteq [0,1]$ whose measure is < 1.
From this argument it follows that B, the unit ball of the Banach lattice L_1 based on $[0,1]$ with Lebesgue measure, is *not* ω-homogeneous, even though it is the unique separable model of T_0. The automorphism group of B has two orbits.

In this section we fix T to be the theory of metric spaces of diameter ≤ 1. Our objective is to explain the following result, and then to say a few more things about this situation.

Theorem

T has a model companion T^*. Moreover:
(1) T^* admits quantifier elimination.
(2) T^* is separably categorical.
(3) The unique separable model of T^* is the Urysohn space $\mathbb{U}$ of diameter 1 (sometimes also called the Urysohn sphere).

To prove T^* exists we need to axiomatize the class of e.c. metric spaces of diameter ≤ 1. The key to doing this and to proving that T^* is separably categorical is the following Lemma.

Lemma

Suppose I is an index set. Consider families $(x_i \mid i \in I)$ and $(y_i \mid i \in I)$ from metric spaces X and Y respectively. Let $(\epsilon_i \mid i \in I)$ be any family of real numbers ≥ 0.
The following are equivalent:

1. *There are a metric space Z and isometries $f\colon X \to Z$ and $g\colon Y \to Z$ satisfying $d(f(x_i), g(y_i)) \leq \epsilon_i$ for all $i \in I$.*
2. *For all $i, j \in I$ we have $\left|d(x_i, x_j) - d(y_i, y_j)\right| \leq \epsilon_i + \epsilon_j$.*

Note that if X and Y have diameter ≤ 1, then we can easily modify Z in condition 1 so that it also has diameter ≤ 1 without changing the statement, simply by truncating its metric at 1.

Proof of Lemma: $(1 \Rightarrow 2)$ is immediate from the triangle inequality. $(2 \Leftarrow 1)$ Easy amalgamation techniques for metric spaces allow us to assume $X = \{x_i \mid i \in I\}$ and $Y = \{y_i \mid i \in I\}$, and we may assume $X \cap Y = \varnothing$. Begin to define a pseudometric d on $X \cup Y$ by taking distances in X and in Y to be the given ones, and to tentatively set the distance between x_i and y_i to be ϵ_i for each i. Then define $d(u, v)$ on $X \cup Y$ to be the infimum of all sums of tentative distances along finite paths joining u to v; allowed edges of the path stay in X or stay in Y or go between x_i and y_i (points with the same index). Condition 2 is used to show that this definition does not change distances in X or in Y. Finally, Z is taken to be the metric space quotient of $(X \cup Y, d)$.

Note: a result with the same essential content as the Lemma above (and essentially the same proof) appears as Proposition 7.1 in the 2008 paper *On subgroups of minimal topological groups* by Vladimir V. Uspenskij in Topology and its Applications 155 (2008), pp. 1580-1606. In that paper only finite index sets are considered and the ϵ_i are all taken to be the same. We need to allow the ϵ_i to vary with i in order to calculate the metric on type spaces over nonempty sets of parameters.

It is now easy to axiomatize the class of e.c. metric spaces of diameter 1. Given a finite sequence $a = a_1, \ldots, a_n$ in a metric space, let $D_a(x)$ be the following quantifier-free formula in the signature of metric spaces:

$$\max_{1 \leq i,j \leq n} \left| d(x_i, x_j) - d(a_i, a_j) \right|$$

Evidently, if $b = b_1, \ldots, b_n$ is another n-tuple from some metric space Y, we have that $D_a^Y(b) = 0$ if and only if the map taking a_i to b_i for $i = 1, \ldots, n$ is isometric.

Theorem

The class of e.c. metric spaces of diameter ≤ 1 is axiomatized by the conditions

$$\sup_x \inf_y \left(D_{a,b}(x, y) \dot{-} D_a(x) \right) = 0$$

where $a, b = a_1, \ldots, a_n, b$ ranges over all finite metric spaces.

Now we know there is a model companion T^* for the theory T of all metric spaces of diameter ≤ 1. Since models of T satisfy the amalgamation property over substructures, we know that T^* admits QE. In this simple signature, every atomic formula is of the form $d(x, y)$ where x, y are variables. Therefore, for each finite tuple a in a metric space X of diameter ≤ 1 we can identify $\mathrm{tp}_X(a)$ with the matrix $\big(d(a_i, a_j)\big)_{i,j}$. Moreover, the Lemma above yields the following formula for the distance between types:

$$d(\mathrm{tp}_X(a), \mathrm{tp}_Y(b)) = \frac{1}{2} \max_{i,j} \left| d(a_i, a_j) - d(b_i, b_j) \right|$$

This proves that for all $n \geq 1$, the type space $\mathrm{S}_n(T^*)$ is compact for the metric topology. Hence T^* is separably categorical.

As above, we let $\mathbb{U}$ denote the unique separable model of T^*, which is the Urysohn space of diameter 1 (= the sphere of radius $\frac{1}{2}$ in the unbounded Urysohn space).

Let $\mathbb{M}$ denote a monster model of T^* (= κ-saturated and κ-homogeneous for some large cardinal κ). We may assume $\mathbb{U} \subseteq \mathbb{M}$.

The approach used above can be applied to the type spaces $\mathsf{S}_n(A)$ for any small subset $A \subseteq \mathbb{M}$.

A 1-type $p \in \mathrm{S}_1(A)$ is realized by some element $b \in \mathbb{M}$. Since T^* admits QE, we may identify p with the function $f_p \colon A \to [0,1]$ given by $f_p(a) = d(a,b)$. Based on this identification, we see that as sets, $S_1(A) = K_{\leq 1}(A)$, the set of Katětov functions with values in $[0,1]$. The identification extends to the topological and metric structure, as well. Recall that on $K_{\leq 1}(A)$ we put the topology of pointwise convergence and the supremum metric.

Proposition

For each metric space A, the identification above is an isomorphism of topometric spaces between $S_1(A)$ and $K_{\leq 1}(A)$. That is, it is a homeomorphism and an isometry.

A new example

Expand the pure language of metric spaces by adding finitely many predicate symbols $P_1, \ldots, P_n$ and for each $i = 1, \ldots, n$ fix a real number $C_i > 0$. Now let T_P denote the theory of all structures M for this signature that satisfy Lipschitz axioms of the following form (one for each i):

$$\big|P_i(x) - P_i(y)\big| \leq \sum_j C_i d(x_j, y_j)$$

A new example

Theorem

T_P has a model companion T_P^*. Moreover:
(1) T_P^* admits quantifier elimination.
(2) T_P^* is separably categorical.
(3) The underlying metric space of the unique separable model of T_P^* is the Urysohn space $\mathbb{U}$ of diameter 1 (i.e., the restriction of T_P^* to the pure metric space signature is T^*) .

Some references

- Itaï Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov, *Model theory for metric structures*, in Lecture Notes series of the London Mathematical Society, No. 350, Cambridge University Press, 2008, 315–427.
  [Treats QE in chapter 13; separable models in chapter 12.]
- Itaï Ben Yaacov and Alexander Usvyatsov, *On d-finiteness in continuous structures*, Fundamenta Mathematicae 194 (2007), 67-88.
  [Treats omitting types and separable models in section 1.]
- C. Ward Henson and José Iovino, *Ultraproducts in analysis*, in Analysis and Logic, London Mathematical Society Lecture Notes Series, vol. 262, 2002, 1–113.
  [Treats atomless L_p spaces in Example 13.18.]