Continuous first-order model theory for metric structures Lecture 1 (of 3)

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Origins

Many classes of (complete) metric structures arising in analysis and geometry are well-behaved model theoretically, although they are not elementary in the classical sense.

Continuous logic is an attempt to apply model-theoretic tools to such classes. It was preceded by:

- Use of an ultraproduct construction in analysis.
- Henson's logic for Banach structures (positive bounded formulas, approximate satisfaction).
- Ben-Yaacov's positive logic and compact abstract theories.
- Chang and Keisler's continuous model theory (1966).
- Łukasiewicz's $[0,1]$-valued logic (only used special connectives).
- Krivine's real-valued logic (only used universal quantifiers).
- ... perhaps others ...

Replace $\{T, F\}$ with $[0, 1]$

- The basic idea is: replace the space of truth values $\{T, F\}$ by a compact interval such as $[0, 1]$.
- Quantifiers $\forall x$ and $\exists x$ are replaced by $\sup_x$ and $\inf_x$.
- Connectives are continuous functions.

Ingredient I: non-logical symbols

- A signature $\mathcal{L}$ consists of function and predicate symbols, as usual.
  - n-ary function symbols: interpreted as functions $M^n \to M$.
  - n-ary predicate symbols: interpreted as functions $M^n \to [0, 1]$.
- $\mathcal{L}$-terms and atomic $\mathcal{L}$-formulas are built inductively as in classical logic.
- $\mathcal{L}$ specifies a modulus of uniform continuity for each function symbol and predicate symbol. (e.g.: 1-Lipshitz.)
- The metric is considered as a (logical) predicate (as equality is used in classical logic).

Example: probability algebras

Probability algebras are Boolean algebras of events in probability spaces; the probability of an event is a value in $[0, 1]$.
The signature of probability algebras is $\mathcal{L} = \{0, 1, \cdot^c, \cap, \cup, \mu\}$.

- 0, 1 are 0-ary function symbols (constant symbols).
- $\cdot^c$ (complement) is a unary function symbol.
- $\cup$, $\cap$ (union, intersection) are binary function symbols.
- μ (probability) is a unary predicate symbol.

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Thus:

- $z, x \cap y^c, x \triangle y$ are terms (values in the algebra).
- $\mu(x), \mu(x \cap y^c)$ are atomic formulas (values in $[0,1]$).

Example: (unit balls of) Banach spaces

The signature of unit balls of Banach spaces is $\mathcal{L} = \{0, c_{r,s}, \| \ \|\}$ where r, s range over pairs of scalars such that $|r| + |s| \leq 1$.

- 0 is a constant symbol.
- $c_{r,s}$ are binary function symbols.
  - Interpret $c_{r,s}(x, y)$ as $rx + sy$.
- $\| \ \|$ is a unary predicate symbol.

Ingredient II: Connectives

- Any continuous function $[0,1]^n \to [0,1]$ is admitted as an n-ary connective.
- That makes syntax uncountable. BUT, a uniformly dense set of connectives is good enough.
- The following connectives generate (countable) uniformly dense families of n-ary connectives (by the $[0,1]$-valued lattice Stone-Weierstrass Theorem on $[0,1]^n$):

$$\neg x := 1 - x; \qquad \tfrac{1}{2}x := x/2; \qquad x \dot{-} y := \max\{x - y, 0\}.$$

- "$\varphi \dot{-} \psi$" replaces "$\psi \to \varphi$".
- Modus Ponens translates to "$\varphi \leq \psi \dot{+} (\varphi \dot{-} \psi)$"; in particular, if $\psi = 0$ and $\varphi \dot{-} \psi = 0$ then $\varphi = 0$.

Ingredient III: Quantifiers

In building formulas:

- We use “$\sup_x \varphi$” and “$\inf_x \varphi$” instead of “$\forall x \varphi$” and “$\exists x \varphi$”.

Structures

Definition

An $\mathcal{L}$-pre-structure is a set M, equipped with a pseudo-metric d^M and uniformly continuous (as specified by $\mathcal{L}$) interpretations f^M, P^M of all symbols $f, P \in \mathcal{L}$.
It is an $\mathcal{L}$-structure if d^M is a complete metric.

- In classical logic $=^M$ is a congruence relation; thus the quotient of M by $=^M$ is well-defined and it cannot be distinguished from M by the logic.
- Similarly, in continuous logic the interpretations of symbols in a pre-structure M are uniformly continuous and M is logically indistinguishable from the completion $\widehat{N}$ of the quotient $N = M/{\sim_d}$. $(a \sim_d b \Longleftrightarrow d(a,b) = 0)$
- We call $\widehat{M/{\sim_d}}$ the completion of the pre-structure M.

Recap: probability algebras

- Let $(\Omega, \mathfrak{B}, \mu)$ be a probability space.
- Let $\mathfrak{B}_0 \leq \mathfrak{B}$ be the ideal of μ-null sets, and $\widetilde{\mathfrak{B}} = \mathfrak{B}/\mathfrak{B}_0$. Then $\widetilde{\mathfrak{B}}$ is a Boolean algebra and μ induces $\widetilde{\mu} \colon \widetilde{\mathfrak{B}} \to [0, 1]$. The pair $(\widetilde{\mathfrak{B}}, \widetilde{\mu})$ is a probability algebra.
- $\widetilde{\mathfrak{B}}$ admits a complete metric: $d(a, b) = \widetilde{\mu}(a \triangle b)$.
- $\widetilde{\mu}$ and the Boolean operations are 1-Lipschitz.

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- $\widetilde{\mathfrak{B}}$ admits a complete metric: $d(a,b) = \widetilde{\mu}(a \triangle b)$.
- $\widetilde{\mu}$ and the Boolean operations are 1-Lipschitz.
- $(\mathfrak{B}, 0, 1, \cap, \cup, \cdot^c, \mu)$ is a pre-structure; $(\widetilde{\mathfrak{B}}, 0, 1, \cap, \cup, \cdot^c, \widetilde{\mu})$ is its completion (in particular, it is a structure).

Semantics

As usual, the notation $\varphi(x_1, \dots, x_n)$ [or simply $\varphi(x)$] means that the free variables of φ are among $x_1, \dots, x_n$, which are distinct. If M is a pre-structure and $a \in M^n$, we define the value $\varphi^M(a) \in [0, 1]$ inductively, in the "obvious way".

- $\big(P(t_1(x), \dots, t_k(x))\big)^{\mathcal{M}}(a) = P^{\mathcal{M}}(t_1^{\mathcal{M}}(a), \dots, t_k^{\mathcal{M}}(a))$;
- $\big(u(\varphi_1(x), \dots, \varphi_k(x))\big)^{\mathcal{M}}(a) = u(\varphi_1^{\mathcal{M}}(a), \dots, \varphi_k^{\mathcal{M}}(a))$;
- $\big(\sup_y \varphi(x, y)\big)^{\mathcal{M}}(a) = \sup\{\varphi^{\mathcal{M}}(a, b) \mid b \in M\}$.
- $\big(\inf_y \varphi(x, y)\big)^{\mathcal{M}}(a) = \inf\{\varphi^{\mathcal{M}}(a, b) \mid b \in M\}$.

Lemma

Each function $\varphi^M \colon M^n \to [0,1]$ *is uniformly continuous, and the modulus of uniform continuity only depends on the signature of* M*.*

Proof.

By induction on syntactic complexity of φ. □

Example

Let $(M, 0, 1, .^c, \cup, \cap, \mu)$ be a probability algebra, and take $\varphi(x)$ to be the formula $\inf_y \left|\mu(x \cap y) - \frac{1}{2}\mu(x)\right|$.

- If $a \in M$ is an atom, then $\varphi^M(a) = \frac{1}{2}\mu(a)$.
- If a has no atoms below it then $\varphi^M(a) = 0$.

Thus a probability algebra M is atomless iff it satisfies the condition

$$\sup_x \inf_y \left|\mu(x \cap y) - \frac{1}{2}\mu(x)\right| = 0$$

Various "elementary" notions

- Elementary equivalence (denoted $M \equiv N$): If M, N are two pre-structures then $M \equiv N$ if $\varphi^M = \varphi^N \in [0, 1]$ for every sentence φ (i.e.: formula without free variables). Equivalently: "$\varphi^M = 0 \iff \varphi^N = 0$ for all sentences φ."
- Elementary extension (denoted $M \preceq N$): This holds if $M \subseteq N$ and $\varphi^M(a) = \varphi^N(a)$ for every formula φ and $a \in M$. It implies $M \equiv N$.

Lemma (Elementary chains)

The union of an elementary chain $M_0 \preceq M_1 \preceq \dots$ is an elementary extension of each M_i.

Caution: by the union of an increasing chain we mean the completion of its set-theoretic union.

Ultraproducts

- $(M_i \mid i \in I)$ are $\mathcal{L}$-structures, $\mathscr{U}$ an ultrafilter on I.
- We let $N_0 = \prod_{i \in I} M_i$ as a set; its members are $(a) = (a_i \mid i \in I)$, $a_i \in M_i$.
- We interpret the symbols:

$$\begin{aligned} f^{N_0}\big((a_i \mid i \in I), \dots\big) &= \big(f^{M_i}(a_i, \dots) \mid i \in I\big) && \in N_0 \\ P^{N_0}\big((a_i \mid i \in I), \dots\big) &= \lim_{i,\mathscr{U}} P^{M_i}(a_i, \dots) && \in [0,1] \end{aligned}$$

- This way N_0 is a pre-structure. We define the ultraproduct to be $\widehat{N_0}$ (the completion) and denote it by $\prod_{i \in I} M_i/\mathscr{U}$.
- The image of $(a) \in N_0$ in $\widehat{N_0}$ is denoted $(a)_{\mathscr{U}}$; note that

$$(a)_{\mathscr{U}} = (b)_{\mathscr{U}} \iff 0 = \lim_{i,\mathscr{U}} d(a_i, b_i) \quad \Big[= d^{N_0}\big((a),(b)\big)\Big].$$

Properties of ultraproducts

- Łoś's Theorem: for every formula $\varphi(x, y, \ldots)$ and elements $(a)_{\mathscr{U}}, (b)_{\mathscr{U}}, \ldots \in N = \prod M_i/\mathscr{U}$:

$$\varphi^N\big((a)_{\mathscr{U}}, (b)_{\mathscr{U}}, \ldots\big) = \lim_{i,\mathscr{U}} \varphi^{M_i}(a_i, b_i, \ldots).$$

- [Easy] $M \equiv N$ (M and N are elementarily equivalent) if and only if M admits an elementary embedding into an ultrapower of N.
- [Deeper: generalising Keisler & Shelah] $M \equiv N$ if and only if M and N have ultrapowers that are isomorphic.

Theories

- A theory T is a set of sentences (closed formulas).
- M is a model of T (written $M \vDash T$) $\iff \varphi^M = 0$ for all $\varphi \in T$ AND M is a structure.
- We sometimes write T as a set of conditions "$\varphi = 0$".
  We may also allow as conditions things of the form "$\varphi \leq r$", "$\varphi \geq r$", "$\varphi = r$", etc.
- If M is any structure then its theory is

$$\mathrm{Th}(M) = \{\varphi \mid \varphi^M = 0\} \quad \Big[\equiv \{ \big|\psi - r\big| \mid \psi^M = r\} \Big].$$

  Theories of this form are called complete (equivalently: complete theories are the maximal satisfiable theories).

Compactness

Theorem (Compactness)

A theory is satisfiable if and only if it is finitely satisfiable.

Note that:

$$T \equiv \{\text{“}\varphi \leq 2^{-n}\text{”} \mid n < \omega \text{ and “}\varphi = 0\text{”} \in T\}.$$

Corollary

Assume that T is approximately finitely satisfiable. Then T is satisfiable.

Löwenheim-Skolem Theorem

Notation: $\| \cdot \|$ denotes the metric density character.
$\mathcal{L}$ is a signature with $\leq \kappa$ symbols.
M is an $\mathcal{L}$-structure and $A \subseteq M$ has $\|A\| \leq \kappa$.

Theorem

There exists an elementary substructure N of M such that $A \subseteq N$ and $\|N\| \leq \kappa$.

Elementary (axiomatizable) classes

A class $\mathcal{C}$ of $\mathcal{L}$-structures is elementary or axiomatizable if there is an $\mathcal{L}$-theory T such that $\mathcal{C}$ is the class of all models of T. When this holds we call T a set of axioms for $\mathcal{C}$.

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Theorem

Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures. Then $\mathcal{C}$ is axiomatizable iff $\mathcal{C}$ is closed under isomorphisms, ultraproducts, and ultraroots.

Here: M is an ultraroot of N if N is isomorphic to some ultrapower of M.

Types (without parameters)

Definition

Let M be an $\mathcal{L}$-structure, $a \in M^n$. Then:

$$\mathrm{tp}^M(a) = \{\varphi(x) \mid \varphi(x) \in \mathcal{L}, \varphi(a)^M = 0\} \equiv \mathrm{Th}(M, a).$$

$\mathrm{S}_n(T)$ is the space of types of n-tuples in models of T. If $p \in \mathrm{S}_n(T)$:

$$\varphi(x)^p = r \iff \left|\varphi(x) - r\right| \in p.$$

- The logic topology on $S_n(T)$ is the minimal topology such that $p \mapsto \varphi^p$ is continuous for all φ.
- This is the analogue of the Stone topology on types in classical logic; it is compact and Hausdorff (but it is NOT totally disconnected, in general).

Closed sets (for the logic topology) in $\mathsf{S}_n(T)$ are the type definable sets. That is, they are exactly the sets of the form

$$\{p \in \mathsf{S}_n(T) \mid \Phi(x) \subseteq p\}$$

where $\Phi(x)$ is any set of formulas in x.

Types (with parameters)

Definition

Let M be a structure, $a \in M^n$, $B \subseteq M$. Then:

$$\operatorname{tp}^M(a/B) = \{\varphi(x,b) \mid \varphi(x,y) \in \mathcal{L}, b \in B^m, \varphi(a,b)^M = 0\}.$$

$\mathrm{S}_n(B)$ is the space of types over B of n-tuples in elementary extensions of M. If $p \in \mathrm{S}_n(B)$, $b \in B$:

$$\varphi(x,b)^p = r \iff \left|\varphi(x,b) - r\right| \in p.$$

The logic topology on $S_n(B)$ is minimal such that $p \mapsto \varphi(x,b)^p$ is continuous for all $\varphi(x,b)$, $b \in B^m$. It is compact and Hausdorff.

Saturated and homogeneous models

Definition

Let κ be a cardinal, M a structure.

- M is κ-saturated if for every $A \subseteq M$ such that $|A| < \kappa$ and every $p \in \mathrm{S}_1(A)$: p is realized in M.
- M is κ-homogeneous if for every $A \subseteq M$ such that $|A| < \kappa$ and every mapping $f\colon A \to M$ that preserves values of formulas in M, f extends to an automorphism of M.

Fact

Let M be any structure and $\mathscr{U}$ a non-principal ultrafilter on $\aleph_0$. Then the ultrapower $M^{\aleph_0}/\mathscr{U}$ is $\aleph_1$-saturated.

Monster models

A monster model of a complete theory T is a model of T that is κ-saturated and κ-homogeneous for some κ that is much larger than any set under consideration.

Fact

- *Every complete theory T has a monster model.*
- *If $\mathbb{M}$ is a monster model for T, then every "small" model of T (i.e., smaller than κ) is isomorphic to some $N \preceq \mathbb{M}$.*
- *If $A \subseteq \mathbb{M}$ is small then $\mathrm{S}_n(A)$ is the set of orbits in $\mathbb{M}^n$ under $\mathrm{Aut}(\mathbb{M}/A)$.*

Thus monster models serve as "universal domains": everything happens inside, and the automorphism group is large enough.

Definable predicates

- We identify a formula $\varphi(x)$ with the (continuous) function $\widetilde{\varphi}\colon \mathrm{S}_n(T) \to [0,1]$ it induces: $p \mapsto \varphi^p$. By the $[0,1]$-valued Stone-Weierstrass Theorem on $\mathrm{S}_n(T)$, these functions are uniformly dense in $C(\mathrm{S}_n(T),[0,1])$.
- An arbitrary continuous function $\psi\colon \mathrm{S}_n(T) \to [0,1]$ is called a definable predicate. It is a uniform limit of functions induced by formulas: $\psi = \lim_{k\to\infty} \widetilde{\varphi_k}$. Its interpretation in $M \vDash T$ is given for $a \in M^n$ by:

$$\psi^M(a) = \lim_k \varphi_k^M(a).$$

Since each φ_k^M is uniformly continuous, so is ψ^M.

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$$\psi^M(a) = \lim_k \varphi_k^M(a).$$

  Since each φ_k^M is uniformly continuous, so is ψ^M.
- The same applies with parameters. Note that a definable predicate $\lim \varphi_k(x, b_k)$ over an infinite set B of parameters may depend on a countably infinite sequence of elements of B.

Metric on types

The topological structure of $\mathrm{S}_n(T)$ is insufficient, by itself. We also need to consider the distance between types:

$$d(p, q) = \inf\{d(a, b) \mid a, b \in M \vDash T \text{ and } M \vDash p(a) \cup q(b)\}.$$

(In case T is incomplete and p, q belong to different completions: $d(p, q) = \inf \varnothing := \infty$.)

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(In case T is incomplete and p, q belong to different completions: $d(p, q) = \inf \varnothing := \infty$.)

The distance between types is always attained as a minimum: to see this,apply compactness to the partial type:

$$p(x) \cup q(y) \cup \{d(x, y) \leq d(p, q) + 2^{-n} \mid n < \omega\}.$$

Type spaces $(\mathrm{S}_n(T), \tau_{\text{logic}}, d)$ are topometric spaces

- If $f\colon \mathrm{S}_n(T) \to [0,1]$ is (topologically) continuous (*i.e.*, f is a definable predicate) then f is (metrically) uniformly continuous.
- In particular, the topology is coarser than the metric topology.
- If $F \subseteq \mathrm{S}_n(T)$ is closed, then so is the set:

$$B(F,r) = \{p \in \mathrm{S}_n(T) \mid d(p,F) \leq r\}.$$

- In particular, $(\mathrm{S}_n(T), d)$ is a complete metric space.
- And: If $F \subseteq \mathrm{S}_n(T)$ is closed and $p \in \mathrm{S}_n(T)$, then there is $q \in F$ such that $d(p,q) = d(p,F)$.

Recall: signature for (unit balls of) Banach spaces

The signature of unit balls of Banach spaces is $\mathcal{L} = \{0, c_{r,s}, \| \ \|\}$ where r, s range over pairs of scalars such that $|r| + |s| \leq 1$.

- 0 is a constant symbol.
- $c_{r,s}$ are binary function symbols.
  - Interpret $c_{r,s}(x, y)$ as $rx + sy$.
- $\| \ \|$ is a unary predicate symbol.

It is not difficult to axiomatize the class of (unit balls) of Banach spaces. The least trivial axioms are the ones expressing properties such as

$$\forall x(\|x\| \leq \tfrac{1}{2} \Rightarrow \exists y(x = \tfrac{1}{2}y))$$

In continuous model theory, the best one can do when trying to express this is

$$\sup_x \min(\tfrac{1}{2} \dot{-} \|x\|, \inf_y \|x - \tfrac{1}{2}y\|) = 0$$

and it takes a small analysis argument, using the other axioms as well as the requirement that structures be metrically complete, to show that this condition does what is desired.

Ultraproducts of Banach spaces

- Let $\mathscr{U}$ be an ultrafilter on an index set I, and
- X_i a Banach space for each $i \in I$, and let
- M_i be the unit ball structure obtained from X_i.

Exercise

The ultraproduct $\prod M_i/\mathscr{U}$ as defined above is the unit ball of the Banach space ultraproduct $\big(X_i \mid i \in I\big)_{\mathscr{U}}$, as introduced by Dacunha-Castelle and Krivine.

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These observations can be used to show that for structures based on Banach spaces, continuous logic and the logic of positive bounded formulas interpreted approximately are equivalent.

Examples of elementary classes of Banach structures

- Hilbert spaces (infinite dimensional).
- L_p Banach lattices (atomless).
- Bands in $L_p(L_q)$ Banach lattices (doubly atomless).
- Nakano spaces $L_{p(\cdot)}(X, \mathscr{B}, \mu)$ with $p(\cdot)$ bounded by k (atomless, with $p(\cdot)$ having essential range $= K \subseteq [1, k]$).
- $C(X)$ spaces, where X is compact, Hausdorff (X is totally disconnected with no isolated points).
- etc. . . .

(All these "parenthesized" examples are complete theories.)

- Itaï Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov, *Model theory for metric structures*, in Lecture Notes series of the London Mathematical Society, No. 350, Cambridge University Press, 2008, 315–427.
  [Gives a general treatment of continuous model theory.]
- Itaï Ben Yaacov and Alexander Usvyatsov, *Continuous first order logic and local stability*, Transactions of the American Mathematical Society 362 (2010), 5213–5259.
  [Gives a general treatment, with local stability.]

Earlier papers: compact abstract theories

- Itaï Ben Yaacov, *Positive model theory and compact abstract theories*, Journal of Mathematical Logic 3, 2003, 85–118.
- Itaï Ben Yaacov, *Simplicity in compact abstract theories*, Journal of Mathematical Logic 3, 2003, 163–191.
- Itaï Ben Yaacov, *Thickness, and a categoric view of type-space functors*, Fundamenta Mathematicae, 179, 2003, 199–224.
- Itaï Ben Yaacov, *Compactness and independence in non first order frameworks*, Bulletin of Symbolic logic, 11, 2005, 28–50.

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- C. Ward Henson and L. C. Moore, *Nonstandard analysis and the theory of Banach spaces*, in Nonstandard Analysis – Recent Developments (Victoria, B.C., 1980), Springer Lecture Notes in Mathematics No. 983, 1983, 27–112.
- C. Ward Henson and José Iovino, *Ultraproducts in analysis*, in Analysis and Logic, London Mathematical Society Lecture Notes Series, vol. 262, 2002, 1–113.

Earlier papers: Ultraproducts in functional analysis

Ultraproducts were introduced into functional analysis by Dacunha-Castelle and Krivine; Heinrich's old survey gives a nice indication of properties preserved under ultraproducts.

Jean Bretagnolle, Didier Dacunha-Castelle, Jean-Louis Krivine, *Lois stables et espaces L^p*, Annals Inst. Henri Poincaré Sect. B (N.S.) 2, 1965/1966, 231–259.

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Stefan Heinrich, *Ultraproducts in Banach space theory*, Journal für die Reine und Angewandte Mathathematik 313, 1980, 72–104.