On distributive subfactor planar algebras

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"Von Neumann Algebras" Trimester Seminar

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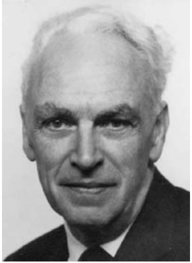
Contents

Inspiration



Øystein Ore (07 October 1899 - 13 August 1968) was a Norwegian mathematician in graph theory, noncommutative ring and number theory.

Ore's theorem for finite groups (1938)

A finite group is cyclic iff its subgroups lattice is distributive.

Let $(N \subseteq M)$ be an irreducible finite index subfactor.

Definition

The subfactor $(N \subseteq M)$ is called distributive if its intermediate subfactors lattice $\mathcal{L}(N \subseteq M)$ is distributive.

Observe that:

- if $(N \subseteq M)$ is maximal then it is distributive.
- a finite group subfactor $(R^G \subseteq R)$ is distributive iff G is cyclic.

Our motivation comes from our own interpretation of:

- the distributive subfactors as "quantum generalization" of the cyclic groups, and so of the natural numbers.

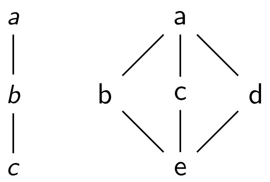
$\mathbb{N}$  $\hbar$

- the theory of distributive subfactors as "quantum arithmetic".

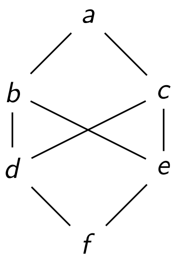
Preliminaries on lattice theory

Definition

A lattice $(L, \wedge, \vee)$ is a partially ordered set (or poset) L in which every two elements a, b have a unique infimum (or meet) $a \wedge b$ and a unique supremum (or join) $a \vee b$.



Lattices



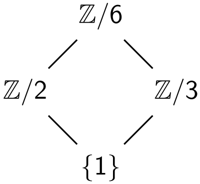
Non-lattice

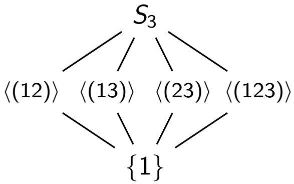
Examples

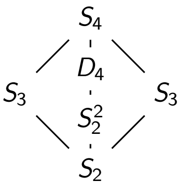
Let G be a finite group. The set of subgroups $K \subseteq G$ is a lattice, denoted by $\mathcal{L}(G)$, ordered by $\subseteq$, with $K_1 \vee K_2 = \langle K_1, K_2 \rangle$ and $K_1 \wedge K_2 = K_1 \cap K_2$.

Definition

A sublattice of $(L, \wedge, \vee)$ is a subset $L' \subseteq L$ such that $(L', \wedge, \vee)$ is also a lattice. Let $a, b \in L$ with $a \leq b$, then the interval $[a, b]$ is the sublattice $\{c \in L \mid a \leq c \leq b\}$.







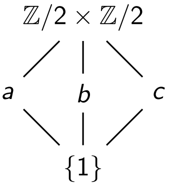
Definition

The lattice $(L, \wedge, \vee)$ is distributive if $\forall a, b, c \in L$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

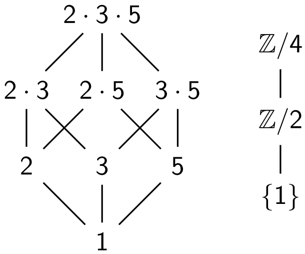
(or equivalently, $\forall a, b, c \in L, \; a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$)

<u>Non-distributive</u>



with $a, b, c \simeq \mathbb{Z}/2$

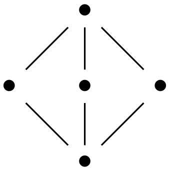
<u>Distributive</u>

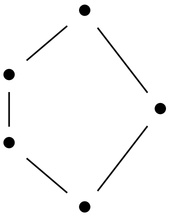


proof: $a \wedge (b \vee c) = a \neq \{1\} = (a \wedge b) \vee (a \wedge c)$

Theorem (Grätzer)

A lattice is distributive if and only if it has no sublattice equivalent to the diamond lattice M_3 or the pentagon lattice N_5, below.





Remark

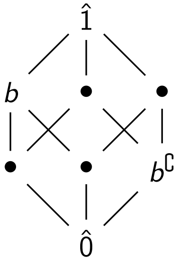
The distributivity is auto-dual and hereditary, i.e. for L distributive, the reversed lattice and every sublattice of L are also distributive.

A finite lattice admits a minimum and a maximum, called $\hat{0}$ and $\hat{1}$.

Definition

The subsets lattice of a set of n elements is the rank n boolean lattice, it's denoted by B_n (see the lattice B_3 below).

$b^{\complement}$ denotes the complementary of b (i.e. $b \wedge b^{\complement} = \hat{0}$, $b \vee b^{\complement} = \hat{1}$).

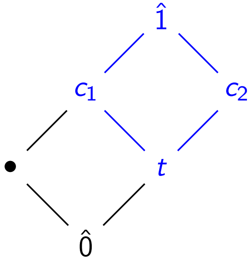


Birkhoff representation theorem (1937)

A finite lattice is distributive iff it embeds into a boolean lattice.

Definitions

A coatom is an element c such that $\forall b \in L,\ c \leq b < \hat{1} \Rightarrow b = c$.
The top interval of a finite lattice is $[t, \hat{1}]$ with t the coatoms meet.



Lemma (R.P. Stanley)

The top interval of a finite distributive lattice is boolean.

Let G be a finite group and H a subgroup.

Definition

The group G is called H-cyclic if $\exists g \in G$ such that $\langle H, g \rangle = G$.

Ore's theorem for intervals (1938)

If the interval $[H, G]$ is distributive, then G is H-cyclic.

Converse false: $\langle S_2, (1234) \rangle = S_4$ but $[S_2, S_4]$ is not distributive.

Lemma 1

If H is a maximal subgroup then G is H-cyclic.

Lemma 2

For $[K, G]$ top interval of $[H, G]$, K-cyclic implies H-cyclic (for G).

So the proof of Ore's theorem reduces to the following Theorem.

Theorem

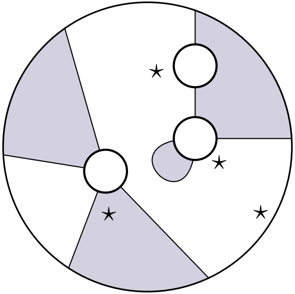
If the interval $[H, G]$ is boolean, then G is H-cyclic.

Proof.

Let M be a coatom of $[H, G]$, and $M^{\complement}$ its complement.
By induction on the rank of the lattice (and Lemma 1), we can assume M and $M^{\complement}$ both H-cyclic, i.e. there are $a, b \in G$ such that $\langle H, a\rangle = M$ and $\langle H, b\rangle = M^{\complement}$. Let $g = ab$ then $a = gb^{-1}$ and $b = a^{-1}g$, so $\langle H, a, g\rangle = \langle H, g, b\rangle = \langle H, a, b\rangle = M \vee M^{\complement} = G$.
Now, $\langle H, g\rangle = \langle H, g\rangle \vee H = \langle H, g\rangle \vee (M \wedge M^{\complement})$ but by distributivity $\langle H, g\rangle \vee (M \wedge M^{\complement}) = (\langle H, g\rangle \vee M\rangle) \wedge (\langle H, g\rangle \vee M^{\complement}\rangle)$.
So $\langle H, g\rangle = \langle H, a, g\rangle \wedge \langle H, g, b\rangle = G$. The result follows. □

A (shaded) planar tangle

- finitely many "input" disks
- one "output" disk
- non-intersecting strings
- giving $2n$ intervals / disk
- one $\star$-marked interval / disk

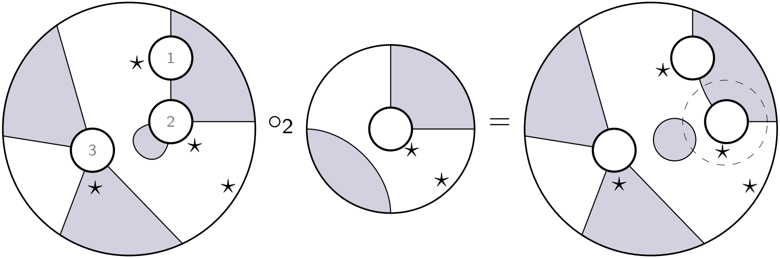


Composition of planar tangles

Put the output disk of one into an input of the other.

- having as many intervals
- same shading of marked intervals
- the $\star$-marked intervals coincide

Finally, remove the coinciding circles.

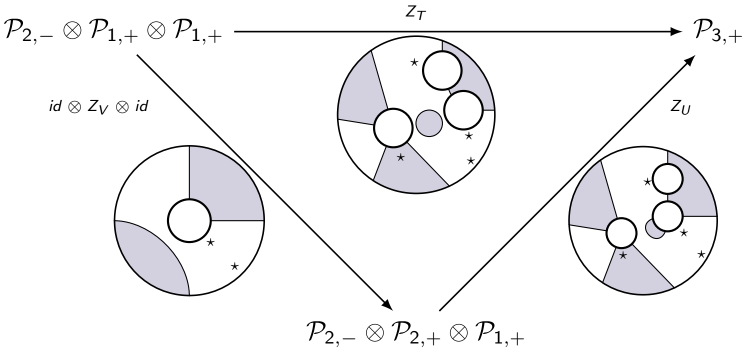


A planar algebra

It is a family of vector spaces $(\mathcal{P}_{n,\pm})_{n\in\mathbb{N}}$, called n-box spaces, on which *acts* every planar tangle T:

$$Z_T : \mathcal{P}_{n_1,\epsilon_1} \otimes \cdots \otimes \mathcal{P}_{n_r,\epsilon_r} \to \mathcal{P}_{n_0,\epsilon_0}$$

respecting the composition (i.e., a planar operad "representation").



A subfactor planar algebra

It is a planar $\star$-algebra $\mathcal{P} = (\mathcal{P}_{n,\pm})_{n\in\mathbb{N}}$ which is:

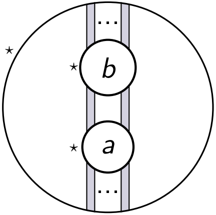
(1) Finite-dimensional: $dim(\mathcal{P}_{n,\pm}) < \infty$

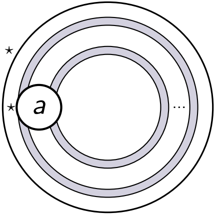
(2) Evaluable: $\mathcal{P}_{0,\pm} = \mathbb{C}$

(3) Spherical: $tr := tr_r = tr_l$

(4) Positive: $\langle a|b\rangle = tr(b^\star a)$ defines an inner product.

By (2) and (3), any closed string counts for the same constant δ.
The subfactor planar algebra $\mathcal{P}$ has the *finite index* δ^2.

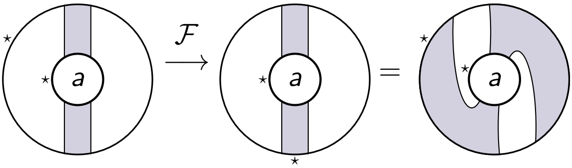
$ab =$ 

$tr_r(a) = \delta^{-n}$ 

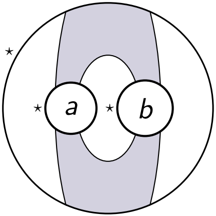
Up to the end, $\mathcal{P}$ will be also assumed *irreducible*, i.e., $\mathcal{P}_{1,\pm} = \mathbb{C}$.

Basic ingredients of the 2-box space

The Fourier transform $\mathcal{F}$ is a 1-click rotation of the outer star.

$\mathcal{P}_{2,+} \ni$  $\in \mathcal{P}_{2,-}$

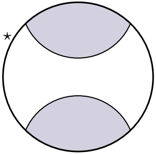
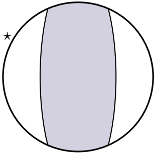
The contragredient $\overline{a} := \mathcal{F}(\mathcal{F}(a))$ corresponds to two 1-clicks.

The coproduct $a * b :=$  $= \mathcal{F}(\mathcal{F}^{-1}(a) \cdot \mathcal{F}^{-1}(b))$.

$\mathcal{F} : (\mathcal{P}_{2,\pm}, +, \cdot, (\cdot)^\star) \to (\mathcal{P}_{2,\mp}, +, *, \overline{(\cdot)}^\star)$ is a vN algebra isom.

A biprojection

It is a projection $b \in \mathcal{P}_{2,+}$ with $\mathcal{F}(b)$ a multiple of a projection.

$e_1 = \frac{1}{\delta}$  $id =$ 

Group-like structures on the 2-box space

group G	identity biprojection id of $\mathcal{P}_{2,+}$
element $g \in G$	minimal projection $u \leq id$
composition gh	coproduct $u * v$
neutral $eg = ge = g$	trivial biprojection $e_1 * u = u * e_1 \sim u$
inverse $g^{-1}g = e$	contragredient $\overline{u} * u \succeq e_1$
subgroup $H \subseteq G$	biprojection $b * b \sim \overline{b} = b = b^* = b^2$

Theorem (Bisch94 merged with Watatani96)

The biprojections ($\leftrightarrow$ inter. subfactors) form a finite lattice $[e_1, id]$.

Coproduct keeps positivity (Liu, 2013)

Let $a, b \in \mathcal{P}_{2,+}$, if $a, b > 0$ then $a * b > 0$.

Biprojection generated (Liu, 2013)

Let $a > 0$ and p_n the range projection of $\sum_{k=1}^{n} a^{*k}$. For N large enough, $p_N =: \langle a \rangle$ is the smallest biprojection $b \succeq a$.

Theorem (P., 2015)

Let $p \in \mathcal{P}_{2,+}$ be a minimal central projection, then there exists $v \leq p$ minimal projection such that $\langle v \rangle = \langle p \rangle$.

Definition

The subfactor planar algebra $\mathcal{P}$ is weakly cyclic (or w-cyclic) if it satisfies one of the following equivalent assertion:

- $\exists u \in \mathcal{P}_{2,+}$ minimal projection such that $\langle u \rangle = id$.
- $\exists p \in \mathcal{P}_{2,+}$ minimal central projection such that $\langle p \rangle = id$.

Theorem (P., 2015)

Let G be a finite group, then $\mathcal{P}(R^G \subseteq R)$ [resp. $\mathcal{P}(R \subseteq R \rtimes G)$] *is w-cyclic iff G is linearly primitive* ($\exists$irr. faithful $\mathbb{C}$-rep.) [resp. cyclic].

Definition

A biprojection $b \in \mathcal{P}_{2,+}$ is lw-cyclic (resp. rw-cyclic) if $\exists u \in \mathcal{P}_{2,+}$ minimal projection such that $\langle u \rangle = b$ (resp. $\langle u, b \rangle = id$).

Intermediate subfactor planar algebras (Landau's Thesis, 1998)

A biprojection $b \in \mathcal{P}_{2,+}$ gives an intermediate subfactor planar algebra on the left $\mathcal{P}(e_1, b)$, and on the right $\mathcal{P}(b, id)$. The idea is to "saturate" every n-box of $\mathcal{P}$ by several copies of b.

Theorem (P., 2015)

A biprojection $b \in \mathcal{P}_{2,+}$ is lw-cyclic (resp. rw-cyclic) iff $\mathcal{P}(e_1, b)$ (resp. $\mathcal{P}(b, id)$) is w-cyclic.

Main result

Definition

Let $b, b' \in [e_1, id]$ with $b \leq b'$ then

- $\ell(b, b')$ is the greatest ℓ with $b < b_1 < \cdots < b_\ell = b'$.
- the index $|b' : b|$ is defined by $tr(b')/tr(b)$ [then $|id : e_1| = \delta^2$].

Definition

An irreducible subfactor planar algebra $\mathcal{P}$ is called boolean (resp. distributive) if its biprojection lattice $[e_1, id]$ is bool. (resp. distr.).

Definition

The Euler totient of $\mathcal{P}$ boolean is defined by

$$\varphi(\mathcal{P}) = \sum_{b \in [e_1, id]} (-1)^{\ell(b,id)} |b : e_1|$$

Definition

For $\mathcal{P}$ distributive, the Euler totient $\varphi(\mathcal{P})$ is defined by

$$|t : e_1| \cdot \varphi(\mathcal{P}(t, id))$$

with $[t, id]$ boolean, the top interval of $[e_1, id]$.

Observation

Let $n = \prod_i p_i^{r_i}$ then $\varphi(\mathcal{P}(R \subseteq R \rtimes \mathbb{Z}/n))$ is equal to

$$\prod_i p_i^{r_i - 1} \cdot \prod_i (p_i - 1)$$

which is the usual Euler's totient $\varphi(n)$ of the natural number n.
So φ extends from natural numbers to distributive subfactors!

The Euler totient extends to any irreducible subfactor planar algebra $\mathcal{P}$ by using the Möbius function on the lattice $[e_1, id]$.

Main Theorem (P., 2016)

Let $\mathcal{P}$ be a distributive subfactor planar algebra. If its Euler totient $\varphi(\mathcal{P})$ is nonzero then $\mathcal{P}$ is w-cyclic.

Lemma (P., 2015)

Let $[t, id]$ the top interval of $[e_1, id]$. If $\mathcal{P}(t, id)$ is w-cyclic, so is $\mathcal{P}$.

Proof.

Let $b_1, \dots, b_n$ be the coatoms of $[e_1, id]$, by assumption and a previous Theorem, $b = \bigwedge_i b_i$ is rw-cyclic, i.e. there is a minimal projection c with $\langle b, c \rangle = id$, but if $\exists i$ such that $c \leq b_i$ then $\langle b, c \rangle \leq b_i$, contradiction, so $\forall i$, $c \not\leq b_i$ and then $\langle c \rangle = id$. □

So the proof of the Main Theorem reduces to the following:

Theorem (P., 2016)

Let $\mathcal{P}$ be a boolean subfactor planar algebra. If its Euler totient $\varphi(\mathcal{P})$ is nonzero then $\mathcal{P}$ is w-cyclic.

Proof Let $p_1, \dots, p_r$ be the minimal central projection of $\mathcal{P}_{2,+}$. Consider the sum

$$S(i) := \sum_{b \in [e_1, id]} (-1)^{\ell(b,id)} tr(bp_i)$$

We can rewrite the sum $S(i)$ as $S_1(i) + S_2(i)$ with

$$S_1(i) := \sum_{b \in [e_1, id];\ bp_i = p_i} (-1)^{\ell(b,id)} tr(bp_i)$$

$$S_2(i) := \sum_{b \in [e_1, id];\ bp_i \neq p_i} (-1)^{\ell(b,id)} tr(bp_i)$$

Let $b_1, \ldots, b_n$ be the coatoms of $[e_1, id]$. Let A_i be the set of coatoms b satisfying $bp_i = p_i$, and B_i the set of coatoms not in A_i. Let ta_i (resp. tb_i) be the meet of all the elements of A_i (resp. B_i).

Claim

For $b \in [e_1, id]$, $bp_i = p_i \Leftrightarrow b \in [ta_i, id]$.

It follows that

$$S_1(i) = \sum_{b \in [ta_i, id]} (-1)^{\ell(b,id)} tr(bp_i) = tr(p_i) \cdot (1-1)^{|A_i|}$$

Now, by the boolean structure, we have

$$[e_1, id] = [ta_i, id] \wedge [tb_i, id]$$

then

$$[e_1, id] \setminus [ta_i, id] = [ta_i, id] \wedge [tb_i, id) = \coprod_{b \in [ta_i, id]} b \wedge [tb_i, id),$$

so with

$$T(i) := \sum_{b \in [tb_i, id)} (-1)^{\ell(b,id)} tr(bp_i)$$

and the fact that $(b \wedge b')p_i = bp_i \wedge b'p_i$, we get that

$$S_2(i) = \sum_{b \in [ta_i, id]} (-1)^{\ell(b,id)} T(i) = T(i) \cdot (1-1)^{|A_i|}$$

Claim

$\mathcal{P}$ is w-cyclic iff $\exists p_i$ minimal central projection, with $|A_i| = 0$.

If $\mathcal{P}$ is not w-cyclic, then $\forall i \ |A_i| \neq 0$, so $S(i) = S_1(i) + S_2(i) = 0$; but $|b : e_1| = tr(b)/tr(e_1)$, $tr(b) = \sum_i tr(bp_i)$ and $tr(e_1) = \delta^{-2}$, so $\varphi(H, G) = \delta^2 \sum_{i=1}^{r} S(i) = 0$; the result follows. $\square$

This result came after two others we get in a previous paper:

Theorem (P., 2015)

Let $\mathcal{P}$ be a distributive subfactor planar algebra. If one of the following (non-equivalent) statements occurs, then $\mathcal{P}$ is w-cyclic.

- *all the biprojections are central.*
- $\sum_i \frac{1}{|id:b_i|} \leq 2$, *with* $b_1, \dots, b_n$ *the coatoms of* $[e_1, id]$.

But the new result using the Euler totient $\varphi(\mathcal{P})$ should be better, because:

Conjecture

Let $\mathcal{P}$ be a boolean subfactor planar algebra. Then $\varphi(\mathcal{P}) \neq 0$.

Remark

It is true for $\mathcal{P}(R \rtimes H \subseteq R \rtimes G)$, because φ is the number of cosets Hg generating G (individually). For the dual case $\mathcal{P}(R^G \subseteq R^H)$, it is checked by GAP at index < 32. The general proof is a *work in progress* with Mamta Balodi.

Now, assume $[e_1, id]$ boolean of rank $n+1$:

Question

Is it true that $\varphi(\mathcal{P}) \geq \phi^n$? (with $\phi = 1.618\dots$, the golden ratio)

Remark

If this lower bound is correct, then it could be optimal because it should be realized by $\mathcal{TLJ}(\sqrt{2}) \otimes \mathcal{TLJ}(\phi)^{\otimes n}$.

Definition

Let $[H, G]$ be a boolean interval of finite groups.
Its Euler totient is defined by

$$\varphi(H, G) := \sum_{K \in [H,G]} (-1)^{\ell(K,G)} |K : H|.$$

Its dual Euler totient is defined by

$$\hat{\varphi}(H, G) := \sum_{K \in [H,G]} (-1)^{\ell(H,K)} |G : K|.$$

Remark

The Euler totient $\varphi(H, G)$ is exactly the number of cosets Hg generating G individually. It is nonzero by Ore's theorem.
Question: Is the dual Euler totient $\hat{\varphi}(H, G)$ also nonzero?

We extend φ and $\hat{\varphi}$ to any interval by using the Möbius function.

Definition

Let W be a representation of a group G, K a subgroup of G, and X a subspace of W. We call the *fixed-point subspace*

$$W^K := \{w \in W \mid kw = w \ , \forall k \in K\}$$

and the *pointwise stabilizer subgroup*

$$G_{(X)} := \{g \in G \mid gx = x \ , \forall x \in X\}$$

Theorem (P., 2016)

If $[H, G]$ is distributive and $\hat{\varphi}(H, G)$ nonzero, then $\exists V$ irreducible complex representation of G such that $G_{(V^H)} = H$.
Moreover, if H is core-free, then G is linearly primitive.

New result in *finite groups theory*, proved by subfactor methods!

Definition

The top boolean Euler length $tbel(\mathcal{P})$ is the minimal length for an ordered chain of biprojections

$$e_1 = b_0 < b_1 < \cdots < b_n = id$$

such that the interval $[b_{\alpha-1}, b_\alpha]$ admits a boolean top with $\varphi \neq 0$.

$tbel(\mathcal{P})$ can be computed directly from the biprojection lattice with indices. It is a purely combinatorial invariant.

Theorem (P., 2016)

Let $\mathcal{P}$ be an irreducible finite index subfactor planar algebra. Then the minimal number of minimal central projections generating the identity biprojection is less than $tbel(\mathcal{P})$, a nontrivial upperbound.

We get analogous theorems in the subfactors, quantum groups and finite groups theories, just by translating the statement above.

For example, a purely combinatorial non-trivial upper-bound for the minimal number of irreducible complex representations generating the left regular representation of a finite group G.

We have found a bridge linking combinatorics and representations, built in the language of subfactor planar algebras!

Work in progress (joint with Mamta Balodi)

We would like to prove that for any boolean subfactor planar algebra $\mathcal{P}$, the Euler totient $\varphi(P)$ is nonzero.

For $\mathcal{P}(R \rtimes H \subseteq R \rtimes G)$ with $[H, G]$ a boolean interval of finite groups, $\varphi(\mathcal{P}) = \varphi(H, G)$ which is is exactly the number of cosets Hg such that $\langle Hg \rangle = G$, and is nonzero by Ore's theorem.

For the dual case, $\mathcal{P}(R^G \subseteq R^H)$, $\varphi(\mathcal{P}) = \hat{\varphi}(H, G)$, the dual Euler totient. It is exactly the Möbius invariant of the bounded coset poset $\hat{P} = \hat{C}(H, G)$ of H in G. Problem: Is it nonzero?

The strategy to prove it is nonzero, is to show that the poset $\hat{P}$ is Cohen-Macaulay (notion by R. Stanley) and the nontrivial reduced Betti number of the order complex $\Delta(P)$ is nonzero.

For so, it is sufficient to prove the existence of, first, a dual EL-labeling on $\hat{P}$, and next, a maximal decreasing chain on it.

Theorem (Balodi, P., 2016)

It is true in the group-complemented case ($\forall K,\ KK^{\complement} = K^{\complement}K$).

Corollary (Balodi, P., 2016)

It is true in general at index < 32.

We are investigating this in general (without index restriction).

After that, we plan to generalize the proof for subfactor planar algebras by extending the notions of coset poset, order complex, Cohen-Macaulay, EL-labeling..., and see how things work.

Preprints

- S. Palcoux, Ore's theorem for cyclic subfactor planar algebras and applications, 2015, arXiv:1505.06649, 50pp
- M. Balodi, S. Palcoux, On boolean intervals of finite groups, 2016, arXiv:1604.06765, 26pp

Thank you for listening.