Analysis of L-convex Function Minimization Algorithms and Application to Auction Theory

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joint work with

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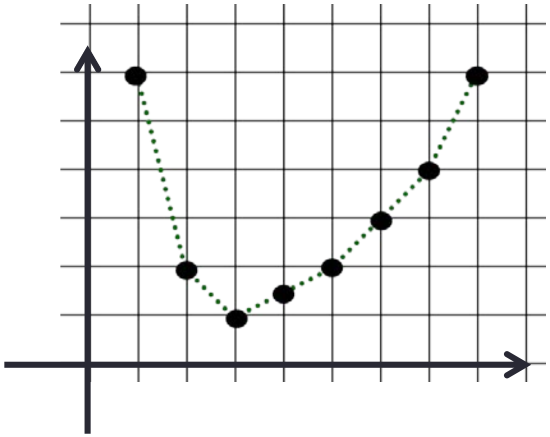
Overview of This Talk

L♮-convex Function:

- "discrete convexity" for functions defined on $\mathbb{Z}^n$



Minimization of L♮-convex Function

- fundamental optimization problem
- various applications
  - auction theory
  - image processing
  - discrete optimization

Our Result 1: Analysis of Minimization Algorithms

- exact bounds for # of iterations

Our Result 2: Application to Auction Theory

L♮-convex Fn in Discrete Convex Analysis

Discrete Convex Analysis (Murota 1998)

- Convexity concepts
  - Two types of discrete convexity: L♮-convex / M♮-convex

- Various properties
  - Conjugacy between L♮-convexity / M♮-convexity

M-L ConjugacyThm: (Murota 1998)
by Legendre transform,
M♮-convex fn $f(x)$ ←→ L♮-convex fn $g(p)$

  - Minimization algorithms for L♮-convex functions

Definition of L♮-convex Fn

continuous fn $g: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex

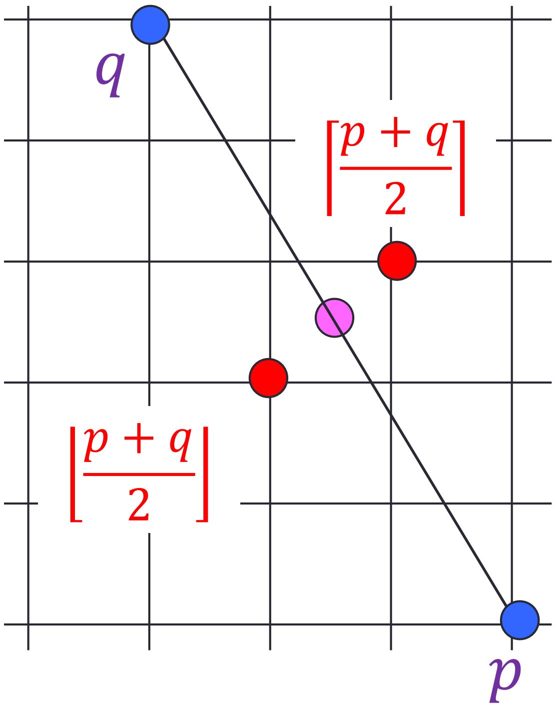
⟺ mid-point convex: $\forall p, q \in \mathbb{R}^n$,

$$g(p) + g(q) \geq 2g\left(\frac{p+q}{2}\right)$$

Def: $g: \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is L♮-convex

⟺ discrete mid-point convex: $\forall p, q \in \mathbb{Z}^n$,

$$g(p) + g(q) \geq g\left(\left\lceil\frac{p+q}{2}\right\rceil\right) + g\left(\left\lfloor\frac{p+q}{2}\right\rfloor\right)$$



Original Definition of L♮-convexity by Submodularity

Def: $g : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is L♮-convex

$\Leftrightarrow$ $\tilde{g} : \mathbb{Z} \times \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is submodular

$$\tilde{g}(p_0, p) = g(p - p_0 \mathbf{1}) \qquad ((p_0, p) \in \mathbb{Z} \times \mathbb{Z}^n)$$

$$\mathbf{1} = (1,1, \ldots, 1)$$

Prop: L♮-convex $\Rightarrow$ submodular on $\mathbb{Z}^n$

L♮-convex fn on $\{0,1\}^n$ $\Leftrightarrow$ submodular set fn

Examples of L♮-convex Fn

- quadratic fn $g(p) = p^T A p$ is L♮-convex

  $\Leftrightarrow$ $a_{ij} \le 0 \ (i \ne j), \quad \sum_j a_{ij} \ge 0$

$$\begin{bmatrix} 4 & & & -1 \\ & 3 & -2 & \\ & -2 & 3 & -1 \\ -1 & & -1 & 5 \end{bmatrix}$$

- range: $g(p) = \max\{p_1, p_2, \ldots, p_n\} - \min\{p_1, p_2, \ldots, p_n\}$

- min-cost tension problem

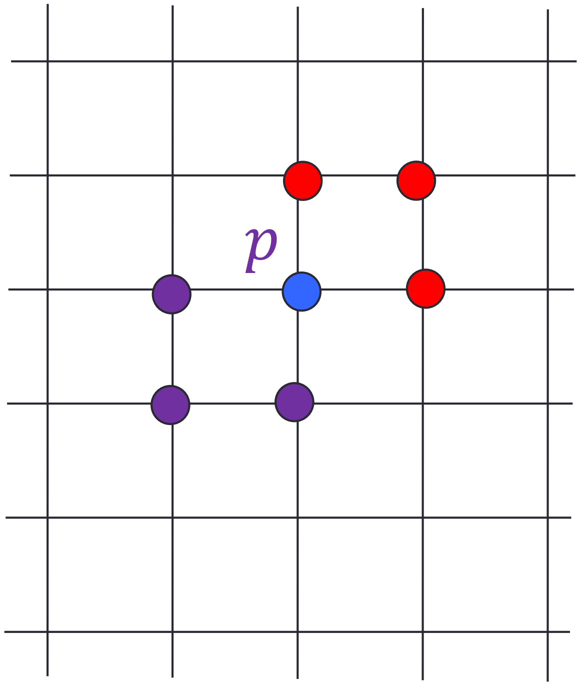
  $g(p) = \sum_{i=1}^{n} \varphi_i(p_i) + \sum_{i,j} \psi_{ij}(p_i - p_j)$ (φ_i, ψ_{ij}: univariate conv)

  - dual of min-cost flow
  - “energy fn” in image processing

Steepest Descent Algorithms for L♮-convex Minimization

Minimization of L♮-convex Fn

$g: \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ --- L♮-convex



Problem: Minimize $g(p)$ for $p \in \mathbb{Z}^n$

Thm (optimality condition): (Murota 1998)

p: global min

⇔ p: local min

$$g(p) \le g(p \pm e_X) \quad (\forall X \subseteq N)$$

$e_X \in \{0,1\}^n$: characteristic vector

$$e_X(v) = \begin{cases} 1 & (v \in X) \\ 0 & (v \in N \setminus X) \end{cases}$$

Algorithms for L♮-convex Minimization

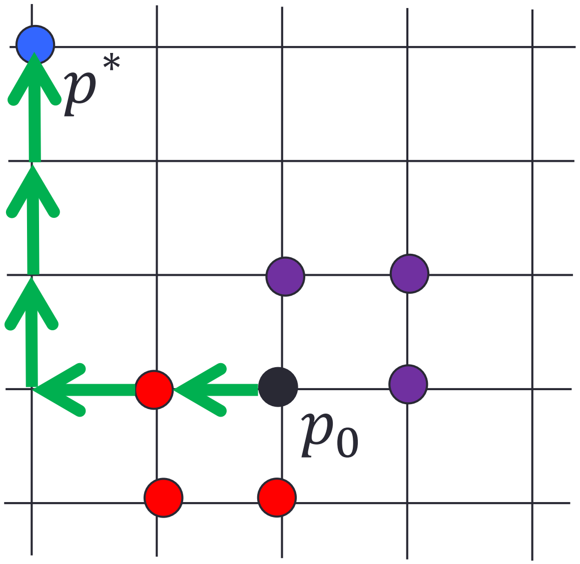
Steepest Descent (Up&Down)

Step 0: $p := p_0$ (initial pt)

Step 1: Take $\delta \in \{+1, -1\}$ & $X \subseteq N$
to minimize $g(p + \delta e_X)$

Step 2: $g(p + \delta e_X) \geq g(p)$ ➔ finish
(current p is opt sol)

Step 3: $p := p + \delta e_X$, Go to Step 1



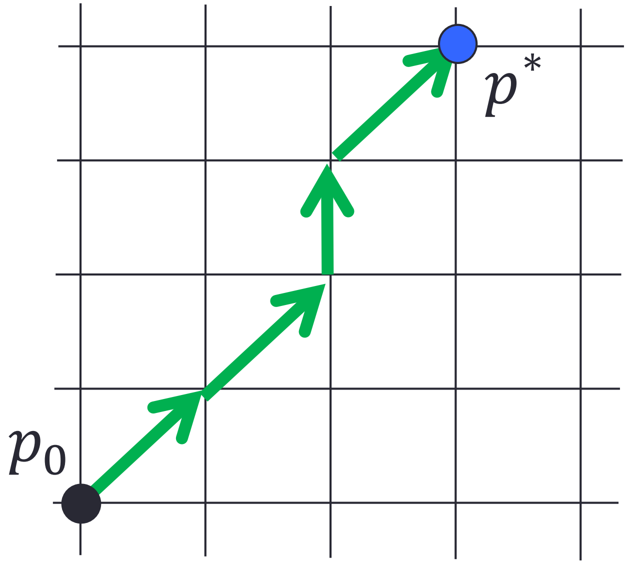
Steepest Descent (Up)

Step 0: $p := p_0$ (sufficiently small)

Step 1: Take $X \subseteq N$
to minimize $g(p + e_X)$

Step 2: $g(p + e_X) \geq g(p)$ ➔ finish
(current p is opt sol)

Step 3: $p := p + e_X$, Go to Step 1



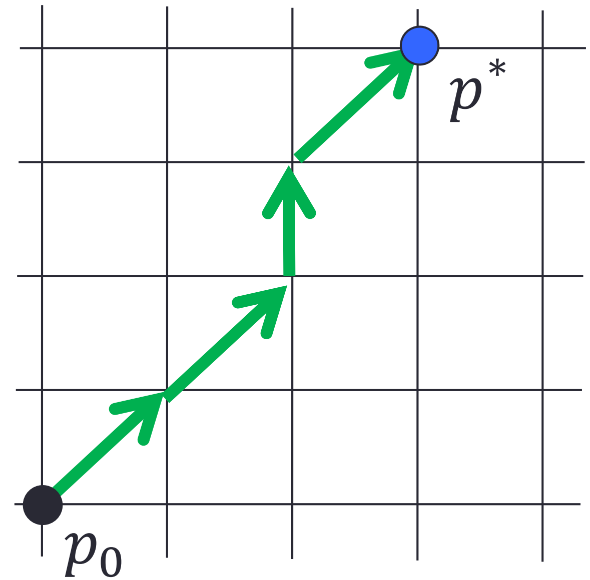
Our Result 1:
Exact Bounds for Number of Iterations

- Previous Results: upper bounds
  by Murota 2004, Kolmogorov-Shioura 2009

- This Talk (Murota-Shioura 2014): exact bounds

Analysis of Number of Iterations (1)

Steepest Descent (Up)
Step 0: $p := p_0$ (sufficiently small)
Step 1: Take $X \subseteq N$
to minimize $g(p + e_X)$
Step 2: $g(p + e_X) \geq g(p)$ ➔ finish
(current p is opt sol)
Step 3: $p := p + e_X$, Go to Step 1



Assumption: $\{p \mid g(p) < +\infty\} \subseteq [-L, +L]^n$

Previous Bounds

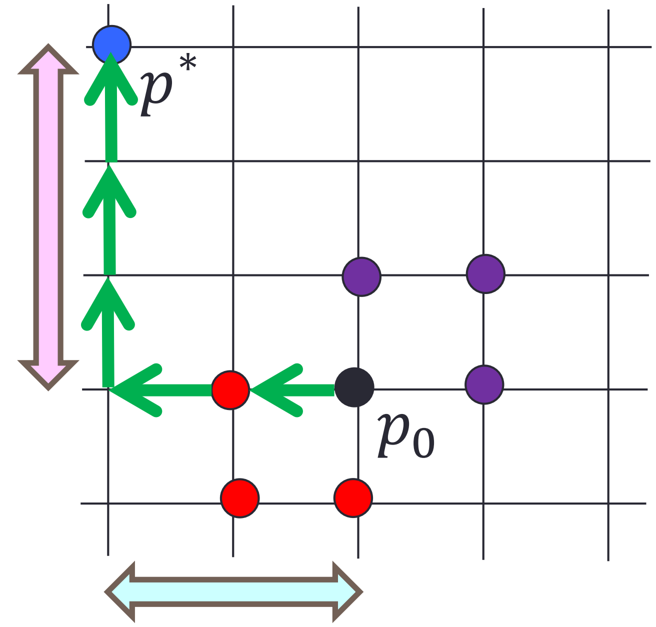
- $\leq 2nL$ (Murota 2004)
- $\leq 2L$ (Kolmogorov-Shioura 2009)

Theorem (this talk):
# of iter. = $\min\{||p^* - p_0||_\infty \mid p^* \in \arg\min g\}$

➔ trajectory of p --- "shortest" path to opt. sol.

Analysis of Number of Iterations (2)

Steepest Descent (Up&Down)
Step 0: $p := p_0$ (initial pt)
Step 1: Take $\delta \in \{+1, -1\}$ & $X \subseteq N$ to minimize $g(p + \delta e_X)$
Step 2: $g(p + \delta e_X) \geq g(p)$ ➔ finish (current p is opt sol)
Step 3: $p := p + \delta e_X$, Go to Step 1



Assumption: $\{p \mid g(p) < +\infty\} \subseteq [-L, +L]^n$

Previous Bounds

- $\leq 4nL$ (Murota 2004)
- $\leq 4L$ (Kolmogorov-Shioura 2009)

Theorem (this talk):

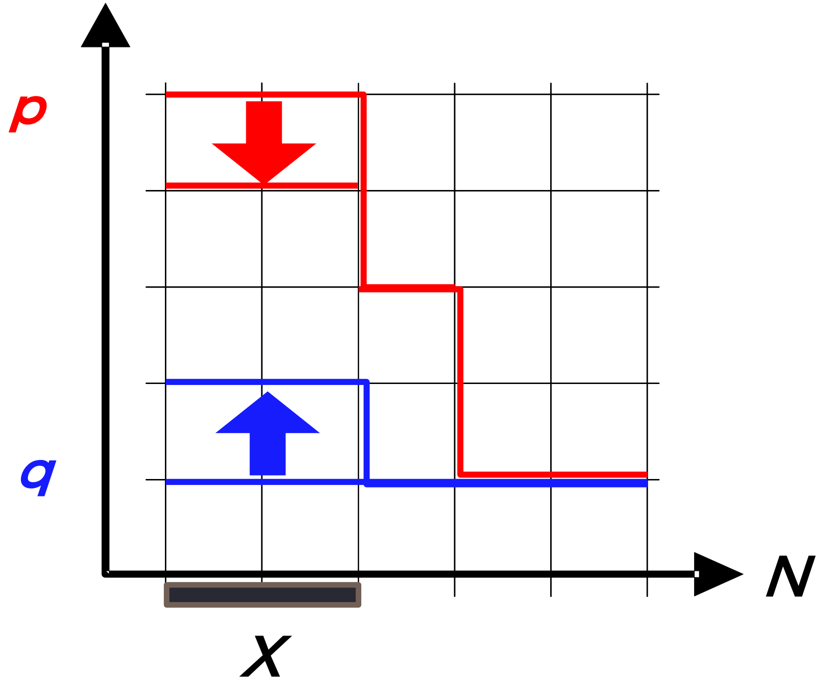
$$\text{\# of iter.} = \min\{ \max[0, \max(p^*(i) - p_0(i))] + \max[0, \max(p_0(i) - p^*(i))] \mid p^* \in \arg\min g \}$$

Useful Property for Precise Analysis

g: L♮-convex function

$$\forall p, q \in \mathbb{Z}^n, \quad X = \arg\max\{p_i - q_i\},$$

$$g(p) + g(q) \geq g(p - e_X) + g(q + e_X)$$



Application to Auction Theory

Auction Setting

items $N = \{1, 2, \dots, n\}$, only one unit available for each

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bidders $M = \{1,2, \dots, m\}$, valuation fn $f_i: 2^N \to \mathbb{Z}_+$

$f_i(S)$ = "value" of item set $S \subseteq N$ for bidder i

- given prices $p = (p_1, p_2, \dots, p_n)$,

bidder want to maximize payoff $f_i(S) - \sum_{i \in S} p_i$

$D_i(p) \equiv \arg\max_{S \subseteq N}\{f_i(S) - \sum_{i \in S} p_i\}$ (demand correspondence)

Def: Walrasian equilibrium prices p^*:

$\exists (S_1^*, S_2^*, \dots, S_m^*)$: partition of N s.t. $S_i^* \in D_i(p^*)$

Gross-Substitutes Condition

Not always: ∃ Walrasian equilibrium

Thm: [Kelso-Crawford (1982), et al.]
f_i: gross-substitutes ➔ ∃ Walrasian equilibrium

(Almost necessary condition for ∃ Walrasian equilibrium)

- Def: gross-substitutes (GS) condition for $f_i\colon 2^N \to \mathbb{Z}_+$

⬅➔ $\forall p \in \mathbb{R}^n$, $q = p + \lambda e_j$,

$\forall X \in D_i(p)$, $\exists Y \in D_i(q)$: $X \setminus \{j\} \subseteq Y$

$$D_i(p) \equiv \arg\max\{\ f_i(S) - \sum_{i \in S} p_i \mid S \subseteq N\}$$

- higher price for iPad,
  more demand for other tablet PCs

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Iterative Auction

- iterative auction: protocol (algorithm)
  for finding equilibrium prices (and allocation)
  - repeatedly update p using bidders' reported information

Step 0. initial prices $p = (p_1, \ldots, p_n)$

Step 1. bidder reports demand correspondence $D_i(p)$

Step 2. If ∃ partition $(S_1^*, S_2^*, \ldots, S_m^*)$ s.t. $S_i^* \in D_i(p)$

➔ stop (p is equilibrium prices)

Step 3. update p. Go to Step 1.

- ascending auction: p increase monotonically

Lyapunov function: Ausubel (2006)

Lyapunov fn: $L(p) = p(N) + \sum_i V_i(p)$

Indirect utility fn $V_i(p) = \max\{f_i(X) - p(X) | X \subseteq N\}$

Prop: (i) L is submodular

(ii) p : minimizer of L ←→ p: equilibrium price

(iii) ∃ integral minimizer of L

Iterative Auctions

= Minimization Algorithms for Lyapunov function

Lyapunov function: Ausubel (2006)

Lyapunov fn: $L(p) = p(N) + \sum_i V_i(p)$

Indirect utility fn $V_i(p) = \max\{f_i(X) - p(X) | X \subseteq N\}$

Ascending Auction

Step 0: $p :=$ sufficiently small integral vector (e.g., $\mathbf{0}$)

Step 1: find $X \subseteq N$ minimizing $L(p + e_X)$

Step 2: $L(p + e_X) \geq L(p)$ ➔ stop (p is equilibrium)

Step 3: $p := p + e_X$, Go to Step 1

$$e_X = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Auction Theory ⇔ Discrete Convex Analysis

Thm: [Kelso-Crawford (1982), et al.]

f_i: gross-substitutes ➔ ∃ Walrasian equilibrium

Def: gross-substitutes (GS) condition for $f_i: 2^N \to \mathbb{Z}_+$

⟺ $\forall p \in \mathbb{R}^n,\ q = p + \lambda e_j,$

$\forall X \in D_i(p),\ \exists Y \in D_i(q):\ X \setminus \{j\} \subseteq Y$

$$D_i(p) \equiv \arg\max\{f_i(S) - \textstyle\sum_{i \in S} p_i \mid S \subseteq N\}$$

Thm: [Fujishige-Yang (2003)]

gross-substitutes ⟺ M♮-concavity

➔ Discrete Convex Analysis

Our Result 2: New View to Iter. Auction

- Valuation fn: Gross substitute ＝ M♮-concave
- Indirect utility, Lyapunov fn ＝ L♮-convex
- Lyapunov fn minimization ＝ L♮-convex minimization
- Iterative Auctions ＝ L♮-conv. minimization algorithms

Discrete Convex Analysis is useful
in Auction Theory

Analysis of Ascending Auction by DCA

Thm: GS for valuation f_i ⇐⇒ M♮-concave (Fujishige-Yang 03)

⇓ M-L Conjugacy Thm (Murota 98)

Thm: indirect utility $V_i(p)$ is L♮-convex
⇐⇒ valuation $f_i(X)$ is GS (Murota-Shioura-Yang 2013)

Cor: Lyapunov fn $L(p)$ is L♮-convex fn (Murota-Shioura-Yang 2013)

Obs: Ascending Auction
= Steepest Descent (Up) for L♮-convex minimization

⇓ Thm on exact bound for Steepest Descent (Up) (this talk)

Cor: $\min\left\{ ||p^* - p_0||_\infty \;\middle|\; p^*: \text{integer equil.},\ p^* \geq p_0 \right\}$

Iterative Auctions from DCA

- Ascending Auction (Ausubel 2006) = Steepest Descent (Up)
- Descending Auction (Ausubel 2006) = Steepest Descent (Down)
  - merit: prices move monotonically
  - demerit: sufficiently small/large initial prices
- Greedy Auction (New) = Steepest Descent (Up&Down)
  - merit: initial prices can be chosen arbitrarily
    # of iterations is small
  - demerit: prices move up & down
- Two-Phase Auction (New)
  - Ascending ➔ Descending
  - merit: initial prices can be chosen arbitrarily
    prices move almost monotonically
  - demerit: larger # of iterations

Two-Phase Auction

Ascending
+ Descending

Step 0: $p :=$ any integral vector
Step A1: find $X \subseteq N$ minimizing $L(p + e_X)$
Step A2: $L(p + e_X) \geq L(p)$ ➔ go to Step D1
Step A3: $p := p + e_X$, Go to Step A1
Step D1: find $Y \subseteq N$ minimizing $L(p - e_Y)$
Step D2: $L(p - e_Y) \geq L(p)$ ➔ stop (p is equilibrium)
Step D3: $p := p - e_Y$, Go to Step D1

- merit
  - any initial vector can be used
  - almost monotone w.r.t. prices
- demerit
  - more iterations than Greedy

Thm: # iters ≦ 3 × (# iters of Greedy)

Summary of 2nd Part

- Valuation fn: Gross substitute ＝ M♮-concave
- Indirect utility, Lyapunov fn ＝ L♮-convex
- Lyapunov fn minimization ＝ L♮-convex minimization
- Iterative Auctions ＝ L♮-conv. minimization algorithms