REPORT JUNIOR TRIMESTER PROGRAM TOPOLOGY

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The authors of this report were the members of one of the groups invited to the Junior Trimester Program on Topology at the Hausdorff Institute for Mathematics from September to December 2016. In this report, intended to be readable by non-specialists, we explain the work we accomplished during this program.

Our project is inspired by Grothendieck's "Esquisse d'un programme" [Gro97]. In this highly influential text, Grothendieck suggests that one should be able to study the absolute Galois group of $\mathbb{Q}$ via its action on objects coming from low-dimensional topology.

The absolute Galois group of $\mathbb{Q}$. Recall that an algebraic number is a complex number that is a solution of a polynomial equation with rational coefficients. These numbers form a subring of the complex numbers. The **absolute Galois group of** $\mathbb{Q}$ is by definition the group of automorphisms of that ring. Very little is known about this group and its understanding is one of the major problems of number theory. It is an uncountable group but paradoxically it is very hard to construct elements of that group. In fact the only non-trivial element with a known explicit description is complex conjugation.

Étale fundamental group and Galois representations. Recall that the **fundamental group** of a based connected topological space X is by definition the group of homotopy classes of loops on that space starting and ending at the base point. The main theorem in covering spaces theory is that the category of covering spaces of X is equivalent to the category of sets equipped with an action of the fundamental group. Grothendieck and his school noticed that although the notion of loops in an algebraic variety makes little sense, there is a purely algebraic description of the covering spaces of an algebraic variety. From this observation one can define the so-called **étale fundamental group** of a complex algebraic variety which is isomorphic to the profinite completion of the topological fundamental group (the profinite completion comes from the fact that only the finite covering spaces can be defined algebraically). In fact the theory uses nothing about the complex numbers and works over any field. From this fact, one obtains the striking result that the absolute Galois group of $\mathbb{Q}$ acts naturally on the étale fundamental group of any complex algebraic variety that is defined over $\mathbb{Q}$. This action can be highly non-trivial. In particular, it was proven by Belyi that this action is faithful on the étale fundamental group of a variety as simple as the projective line with the points 0 1 and ∞ removed.

Moduli spaces of Riemann surfaces of genus zero. The group Γ_0^n is defined as the mapping class group of a Riemann surface of genus zero with n fixed boundary components. Its classifying space $B\Gamma_0^n$ can be interpreted as the classifying space for bundles of such surfaces. It turns out that this space can be given a purely algebraic description : there exists an algebraic variety defined over $\mathbb{Q}$ whose topological space of complex points has the homotopy type of $B\Gamma_0^n$. From this observation and the previous paragraph, one gets an action of the absolute Galois group of $\mathbb{Q}$ on the profinite completion of the group Γ_0^n.

Operads and cyclic operads. The collection of spaces $\{B\Gamma_0^n\}_{n\in\mathbb{N}}$ has a rich algebraic structure. For any pair $(i, j) \in \{1, \dots, n\} \times \{1, \dots, m\}$, there is a gluing map

$$g_{i,j} : B\Gamma_0^n \times B\Gamma_0^m \to B\Gamma_0^{n+m-2}$$

that can be geometrically interpreted as the gluing of a genus zero Riemann surface with n-boundary components to one with m boundary components along the boundary components labelled i and j. The result of this operation is a Riemann surface of genus zero with $n+m-2$ boundary components. These operations satisfy a form of associativity as well as a compatibility with the obvious symmetric group actions on the spaces $B\Gamma_0^n$. Such an algebraic structure is called a **cyclic operad**. If one restricts to the gluing maps $g_{i,j}$ with $j=1$ and $i \neq 1$, one gets the structure of an **operad**. This operad is homotopy equivalent to a famous operad called **the framed little disks operad**.

Our work. As we said, the profinite completion of the mapping class groups Γ_0^n supports an action of the absolute Galois group of $\mathbf{Q}$. In the paper [BHR17] written for the most part during our stay at the HIM, we prove that this action is compatible with the operadic gluing maps $g_{i,j}$ with $j=1$ and $i \neq 1$ defined in the previous paragraph and defines a faithful action of the absolute Galois group of $\mathbb{Q}$ on the profinite completion of the framed little disks operad. Even though the spaces $B\Gamma_0^n$ have the homotopy type of algebraic varieties, it is not at all clear that the gluing maps are homotopic to algebraic maps which makes our result not obvious. In fact our method for proving this result it to completely compute the group of homotopy automorphisms of the operad of profinitely completed moduli spaces $\{B\Gamma_0^n\}_{n\in\mathbb{N}}$. We show that the resulting group is the so-called **profinite Grothendieck-Teichmüller group**. This group was introduced by Ihara in [Iha94] as an approximation of the absolute Galois group of $\mathbb{Q}$ coming from Belyi's theorem and geometric considerations. In particular Ihara proves that the absolute Galois group of $\mathbb{Q}$ injects into the Grothendieck-Teichmüller group.

Furture work. As explained in the previous paragraph, we computed the automorphisms of the operad of moduli spaces as opposed to the richer structure of cyclic operad. We intend to compute the automorphisms of the cyclic operad in future work. We conjecture that the resulting group will again be the Grothendieck-Teichmüller group. A much more ambitious project would be to compute the automorphisms of the **modular operad** of moduli spaces. This is a very complicated object that encodes all the possible gluing of Riemann surfaces along boundary components without any restriction on the genus.

References

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[Iha94] Y. Ihara, *On the embedding of* $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ *into GT*, Dessins d'enfants (Luminy, 1993), London Math. Soc. Lecture Note Ser., vol. 200, Cambridge Univ. Press, Cambridge, 1994.