# FINAL REPORT - THE FARRELL–JONES CONJECTURE

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We participated in the Junior Hausdorff Trimester Program *Topology* from September to December 2016. Our project title was "The Farrell–Jones conjecture". Daniel Kasprowski, Mark Ullmann and Christoph Winges stayed for the whole program while Jianchao Wu stayed from October 16 until September 20.

**The Farrell–Jones conjecture.** The Farrell-Jones conjecture is an important conjecture, connecting algebra, algebraic topology and geometric topology. It implies many other important conjectures, including the Borel conjecture about the topological rigidity of closed aspherical manifolds [BL12a], the Bass conjecture, the Novikov conjecture and the Kaplansky conjecture, see [BLR08c].

The Farrell–Jones conjecture was first formulated in [FJ93]. It has been proved for a large class of groups, for example hyperbolic and CAT(0)–groups, see Bartels and Lück [BL12a, BL12b], Bartels, Lück and Reich [BLR08a, BLR08b] and Wegner [Weg12], virtually solvable groups, see Wegner [Weg15], and lattices in virtually connected Lie groups, see Bartels, Farrell and Lück [BFL14] and Kammeyer, Lück and Rüping [KLR16]. Recently, Bartels [Bar17] showed that the Farrell–Jones conjecture holds for groups which are relatively hyperbolic to groups satisfying the conjecture.

**Seminar.** We organized a weekly learning seminar on the Farrell–Jones conjecture. Some of the talks were given by ourselves while others were given by topologists from Bonn. We also participated in the Trimester Seminar organized together by all groups of the Junior Trimester Program.

**Conference.** From October 24 to October 28 we organized a Workshop on the Farrell–Jones conjecture with 11 invited lectures and 45 registered participants.

**The A–theoretic Farrell–Jones conjecture.** While the Farrell–Jones conjecture is often only formulated for algebraic $K$–theory and $L$–theory, it can also be formulated for Waldhausen's $A$–theory. Before the workshop, in [UW], Ullmann and Winges developed a setting analogous to the linear case to prove the following:

**Theorem** ([UW, Theorem 1.1])**.** *Let $G$ be a virtually poly-$\mathbb{Z}$–group. Then $G$ satisfies the $A$–theoretic Farrell–Jones conjecture.*

During the program Ullmann and Winges worked with Nils-Edvin Enkelmann, Malte Pieper and Wolfgang Lück from Bonn to extend this theorem to hyperbolic and CAT(0)–groups [ELP$^+$].

Kasprowski, Ullmann and Winges also collaborated with Christian Wegner to prove the $A$–theoretic Farrell–Jones conjecture for solvable groups [KUWW].

**Injectivity of the assembly map.** The Farrell–Jones conjecture implies that the assembly maps for the family of finite subgroups are split injective. In $L$–theory, the rational injectivity of the assembly map implies the Novikov conjecture. The integral split injectivity of the assembly map is called the *generalized integral Novikov conjecture*. It has strong ties with the Baum-Connes conjecture for operator $K$–theory.

In [Kas16, Kas15], the generalized integral Novikov conjecture for all subgroups $G$ of virtually connected Lie groups and linear groups over commutative rings was proved under the assumption that there exists a finite dimensional model for $E_{\mathcal{F}in}G$. As for the Farrell–Jones conjecture it makes sense to ask whether these results hold in the $A$–theoretic setting.

Answering this question is part of an on-going collaboration of Kasprowski and Winges with Ulrich Bunke and Alexander Engel. The aim of this project is to extend the setting of motivic coarse homology by Bunke and Engel [BE] to an equivariant version and give suitable conditions for a coarse homology theory that imply split injectivity of the assembly map in this coarse homology theory. Many important discussion on this project happened while Engel and Bunke visited us at the Hausdorff Research Institut during the Junior Trimester Program. More specifically, we expect the following preprints to become available on the arXiv in the near future:

(1) [BEKWb] provides the necessary generalization of [BE] to the equivariant setting and introduces coarse algebraic $K$–homology as the main example to which the general theory applies.
(2) Building on the basic theory developed in [BEKWb], we set up an axiomatic framework to prove injectivity in [BEKWc]. The main result formulates a split-injectivity result for the forget-control map associated to an equivariant coarse homology theory under appropriate assumptions on the theory and the group under consideration. We also show that algebraic $K$–homology fits into this framework.
(3) The third preprint [BEKWa] shows that the main result of [BEKWc] can be applied to groups satisfying a suitable version of finite decomposition complexity.

**Other induction theorems.** A crucial ingredient for the proof of the main theorem in [UW] is an $A$–theoretic analog of Swan's induction theorem. This analog can be derived from a difficult theorem of Oliver about the existence of fixed-point free actions on disks. Akhil Mathew from the group "Interactions between Goodwillie calculus, chromatic methods, and unstable homotopy theory" and Winges have been able to obtain an alternative proof as well as variations of the induction theorem. Instead of Smith theory, the proof relies on an induction result for the $K$–theory of Green functors in conjunction with general facts from equivariant stable homotopy theory. The preprint [MW] will hopefully be available soon.

## References

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