Tangents and dimensions
of
metric spaces

Tapio Rajala
University of Jyväskylä
tapio.m.rajala@jyu.fi

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Tangents

Definition

Let (X, d) be a metric space and $x_0 \in X$. We denote by

$$\mathrm{Tan}(X, x_0)$$

the set of all possible pointed metric spaces (Y, d_Y, y_0) such that there exists a sequence $\lambda_i \to 0$ along which

$$(X, \frac{1}{\lambda_i}\mathrm{d}, x_0) \to (Y, \mathrm{d}_Y, y_0)$$

in the pointed Gromov-Hausdorff convergence.

Tangents

The pointed Gromov-Hausdorff topology can be metrized with the distance

$$\mathrm{D}((X, \mathrm{d}_X, x_0), (Y, \mathrm{d}_Y, y_0)) :=$$

$$\frac{1}{2} \wedge \inf \Big\{ \epsilon > 0 \,\Big|\, \exists \text{ an extension d of the distances on } X \sqcup Y) : \\ \mathrm{d}(x_0, y_0) \leq \epsilon, B_X(x_0, 1/\epsilon) \subset B_{X \sqcup Y}(Y, \epsilon), \\ B_Y(y_0, 1/\epsilon) \subset B_{X \sqcup Y}(X, \epsilon) \Big\}$$

Doubling metric spaces

Definition

A metric space (X, d) is (metric) doubling if there exists a constant $C > 0$ such that for any $z \in X$ and $r > 0$ the ball $B(z, r)$ can be covered by C balls of radius $r/2$.

The set $\mathrm{Tan}(X, x_0)$ is nonempty for any doubling metric space (X, d) and point $x_0 \in X$.

Definition

A nonzero locally finite measure $\mathfrak{m}$ on a metric space (X, d) is doubling if there exists a constant $C > 0$ such that

$$\mathfrak{m}(B(z, 2r)) \leq C\mathfrak{m}(B(z, r))$$

for all $z \in X$ and $r > 0$.

Doubling metric spaces

By a simple volume-argument:

$$\exists \text{ doubling measure} \quad \Longrightarrow \quad \text{metric doubling}$$

In complete metric spaces the converse also holds. A doubling measure can be constructed for example by distributing mass on nested “cubes” in the space.

Notice that a doubling metric space can support many different doubling measures with different properties. For example, the constructed measure can have arbitrarily small (positive) Hausdorff dimension (Tukia, Wu) or even arbitrarily small packing dimension (Käenmäki-R.-Suomala).

Doubling metric spaces

A side remark: The size of sets in a doubling metric space can be studied with all the doubling measures of the space at the same time. A set is called *thin* if it has zero measure for every doubling measure and *fat* if it has positive measure for every doubling measure.

For example

- porous sets are thin
- in $\mathbb{R}^n$ there exists a rectifiable curve that is not thin. (Garnett-Killip-Schul)
- exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ whose graph is not thin. (R.-Ojala)

Thin and fat sets are connected with the analysis of quasisymmetric maps. (Heinonen,Staples-Ward,Wu,Buckley-Hanson-MacManus,Ojala-R.-Suomala,...)

Dimension of doubling metric spaces

Iterating the doubling condition we see that $B(z,r)$ can be covered by C^n balls of radius $2^{-n}r$. Taking $n \to \infty$ leads to

Definition

The Assouad dimension of a metric space (X, d), $\dim_A(X)$, is the infimum of all numbers $\beta > 0$ with the property that there exists $C > 1$ such that, for every $\epsilon > 0$, every Ball of radius r can be covered by using no more than $C\epsilon^{-\beta}$ balls of radius ϵr.

By distributing the mass evenly enough in a complete doubling metric space, we can obtain for all $\alpha > \dim_A(X)$, an α-homogeneous measure $\mathfrak{m}$ on X:

$$\frac{\mathfrak{m}(B(x,r))}{\mathfrak{m}(B(x,s))} \geq C^{-1}\left(\frac{r}{s}\right)^{\alpha}.$$

(Vol'berg-Konyagin,Luukkainen-Saksman)

Typical tangents

There exists a measure in $\mathbb{R}^n$ that has every Radon measure as its tangent measure at almost every point (O'Neil). Similarly one can construct a doubling metric space such that it has at almost every point (w.r.t. a doubling measure) all doubling metric spaces with a given constant $C > 0$ as tangent spaces. One can also construct a compact metric space that has all compact metric spaces as a tangent space at every point. (Chen-Rossi)

Moreover, a generic measure in the Baire category sense has all possible tangents at almost every point of its support. (O'Neil, Sahlsten)

Unique tangents

What if we are in the opposite case where there is only one tangent space at every point?

Theorem (Le Donne)

Supppose that for (X, d) $\mathrm{Tan}(X, x_0) = \{T_{x_0}X\}$ *for every* $x_0 \in X$. *Then*

1. $T_{x_0}X$ *admits dilations for every* $x_0 \in X$.
2. *If* (X, d) *is doubling, then* $T_{x_0}X$ *is homogeneous for every* x_0 *outside a thin set.*
3. *If* (X, d) *is doubling and geodesic, then* $T_{x_0}X$ *is a subFinsler Carnot group for every* x_0 *outside a thin set.*

Unique tangents

Proof.

1. Cosider a sequence $\lambda_i \to 0$ such that $(X, \frac{1}{\lambda_i}\mathrm{d}, x_0) \to T_{x_0}$. Then for any $\delta > 0$ we have $(X, \frac{\delta}{\delta_i}\mathrm{d}/, x_0) \to \delta T_{x_0} = T_{x_0}$.

2. Follows from the iterated tangents result of Preiss:
For almost every $x_0 \in X$ (w.r.t. a doubling measure on X) and for every $(Y, \mathrm{d}_Y, y_0) \in \mathrm{Tan}(X, x_0)$ and $y_1 \in Y$

$$\mathrm{Tan}(Y, y_1) \subset \mathrm{Tan}(X, x_0).$$

3. By results of Gleason, Montgomery, Zippin and Berestovskiĭ T_{x_0} is a subFinsler homogeneous manifold. By a result of Mitchell the tangents of equiregular subFinsler manifolds, like T_{x_0} in our case, are subFinsler Carnot groups. Since T_{x_0} admits dilations, it equals its tangent.

□

Unique tangents

Supposing there is only one tangent space at every point. What can we say about the space itself?

Not much: For example there exist n-regular spaces admitting a local Poincaré inequality with $\mathrm{Tan}(X, x_0) = \{\mathbb{R}^n\}$ for almost every $x_0 \in X$, but still having no manifold points. (Hanson-Heinonen)

Let's assume more.

Uniformly close tangents

Definition

We say that a metric space (X, d) has uniformly close tangents (in $K \subset X$) if

$$\lim_{\lambda \to 0} \mathrm{D}((X, \mathrm{d}/\lambda, z), \mathrm{Tan}(X, z)) = 0$$

uniformly in $z \in K$.

For example, if $\mathrm{Tan}(X, x_0) = \{\mathbb{R}^n\}$ for every $x_0 \in X$ and if X has uniformly close tangents, X are called *Reifenberg vanishing flat*.

Uniformly close tangents

Theorem (Le Donne-R.)

Let (X, d) *have a single tangent space at every point and suppose the convergence to the tangents is (locally) uniform. Then*

$$\dim(X) = \sup_{z \in X} \dim(T_z X),$$

(locally), where dim *is the Assouad dimension, or the Nagata dimension.*

Notice that, we always have $\dim(T_z X) \leq \dim(X)$ for any $T_z X \in \mathrm{Tan}(X, z)$ and $z \in X$. (Not true for example for the Hausdorff dimension.)

Corollary

The Nagata dimension of any bounded open nonempty subset of an equiregular subRiemannian manifold equals the topological dimension of the manifold.

Nagata dimension

Definition

The Nagata dimension of a space is the infimum over all integers n with the property that there exists a constant $c > 0$ such that, for all $s > 0$, the metric space admits an s-bounded cover of the form $\mathcal{B} = \cup_{k=0}^{n} \mathcal{B}_k$ where each $\mathcal{B}_k$ is cs-separated.

Theorem (Le Donne-R.)

$\dim_N(X) \leq \dim_A(X)$.

Nagata-dimension can be used to characterize Lipschitz extendability properties of mappings between metric spaces. (Lang-Schlichenmaier)
Nagata-dimension gives an upperbound on the dimension of metric currents the space can support. (Züst)

Geodesics

Definition

A metric space (X, d) is called a *length space* if

$$\mathrm{d}(x, y) = \inf_{\gamma \text{ joining } x \text{ to } y} \ell(\gamma).$$

The space (X, d) is called *geodesic* if the infimum above is always attained by some curve γ.

- A GH-limit of geodesic spaces is not necessarily geodesic.
- It will always be a length space.
- It will be geodesic under local compactness assumptions.

biLipschitz curves and tangents

- Assuming (X, d) is doubling, the blow-up of a geodesic will be (a possibly half-)infinite line-segment.
- The blow-up of a biLipschitz curve $\gamma \in \Gamma(X)$,

$$\gamma \colon \mathrm{Dom}(\gamma) \to X$$

is an infinite line-segment at $\mathcal{H}^1$-a.e. point in $\mathrm{Dom}(\gamma)$.

Lipschitz differentiability spaces

Definition

A metric measure space $(X, \mathrm{d}, \mathfrak{m})$ is called a *Lipschitz differentiability space* if there exists a countable decomposition of X into charts such that every Lipschitz function $g\colon X \to \mathbb{R}$ is differentiable at $\mathfrak{m}$-almost every point of a chart.

A Borel set $U \subset X$ and a Lipschitz map $\varphi\colon X \to \mathbb{R}^n$, $n \in \mathbb{N}$ form a chart (of dimension n). A Lipschitz function g is differentiable at $x_0 \in U$ with respect to (U, φ) if there exists a unique $Dg(x_0) \in \mathbb{R}^n$ such that

$$\limsup_{x \to x_0} \frac{|g(x) - g(x_0) - Dg(x_0) \cdot (\varphi(x) - \varphi(x_0))|}{d(x, x_0)} = 0.$$

Lipschitz differentiability spaces

Theorem (Cheeger)

Local Poincaré inequality and doubling measure $\Longrightarrow$ Lipschitz differentiability space.

Definition

Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space, $\mathbb{P}$ a Borel probability measure on $\Gamma(X)$ and, for each $\gamma \in \Gamma(X)$, let $\mathfrak{m}_\gamma$ be a Borel measure on X that is absolutely continuous with respect to $\mathcal{H}^1|_{\mathrm{Im}\gamma}$. For a measurable $A \subset X$ we say that the $(\mathbb{P}, \{\mathfrak{m}_\gamma\})$ is an Alberti representation of $\mathfrak{m}|_A$ if for every Borel $Y \subset A$ the function $\gamma \mapsto \mathfrak{m}_\gamma(Y)$ is Borel measurable and

$$\mathfrak{m}(Y) = \int_{\Gamma(X)} \mathfrak{m}_\gamma(Y) \, \mathrm{d}\mathbb{P}(\gamma).$$

Lipschitz differentiability spaces

Theorem (Bate)

An n-dimensional chart (U, φ) in a Lipschitz differentiability space gives n φ-independent Alberti representations (after a countable decomposition of U).
Consequently, for almost every $x \in U$ there exist $\gamma_1^x, \ldots, \gamma_n^x \in \Gamma(X)$ where $(\gamma_i^x)^{-1}(x) = 0$ is a density point of $(\gamma_i^x)^{-1}(U)$ and $(\varphi \circ \gamma_i^x)'(0)$ are linearly independent.

Corollary

Let (U, φ) be an n-dimensional chart in a Lipschitz differentiability space. Then for $\mathfrak{m}$-almost every $x \in U$ any element in $(Y, \mathrm{d}_Y, y) \in \mathrm{Tan}(X, x)$ contains n disjoint (outside y) isometric copies of $\mathbb{R}$ (in independent directions).

Lipschitz differentiability spaces

Theorem (David)

If $(X, \mathrm{d}), \mathfrak{m}$ is Ahlfors n-regular Lipschitz differentiability space with an n-dimensional chart (U, φ), then for $\mathcal{H}^n$-almost every point $x_0 \in U$ each tangent of X at x_0 is uniformly rectifiable: there exist constants $\alpha, \beta > 0$ sucht that for every ball B with center x_0 there exist a set $E \subset B$ with $\mathcal{H}^n(E) \geq \alpha \mathcal{H}^n(B)$ and a β-biLipschitz mapping $f : E \to \mathbb{R}^n$.
Consequently, at almost every point of U there is a tangent of X that is biLipschitz equivalent to $\mathbb{R}^n$.

Notice that an Ahlfors n-regular Lipschitz differentiability space with an n-dimensional chart (U, φ) can still have many "holes", i.e. it need not be locally homeomorphic to an open subset of $\mathbb{R}^n$.

Lipschitz differentiability spaces

If the dimension of the chart in an Ahlfors regular Lipschitz differentiability space is strictly smaller than the dimension of the space, the local structure can be wilder. For example in the Heisenberg group:

- The dimension of the (single) chart is 2.
- The topological dimension and Nagata dimension are 3.
- The Assouad dimension is 4.

There also exist Ahlfors s-regular Lipschitz differentiability spaces with s non-integer. (Laakso)

Spaces with Ricci curvature lower bounds

$RCD(0, N)$-spaces have the splitting property: an isometrically embedded $\mathbb{R}$ splits off leaving a $RCD(0, N-1)$-space. (Gigli)

Therefore at $\mathfrak{m}$-a.e. point of an n dimensional chart of an $RCD(K, N)$-space every tangent splits off $\mathbb{R}^n$. (Cavalletti-R.) Moreover, by the idea of iterated tangents by Preiss, almost everywhere $\mathbb{R}^n$ is one of the tangents. (Gigli-Mondino-R.)

Moreover, an $RCD(K, N)$-space can be decomposed (up to a set of measure zero) to countably many subsets E_j each of which is biLipschitz equivalent to a subset of $\mathbb{R}^{n_j}$. (Mondino-Naber) Therefore, almost everywhere $\mathbb{R}^n$ is the unique tangent (with n possibly depending on the point).

Spaces with Ricci curvature lower bounds

A less restrictive definition:

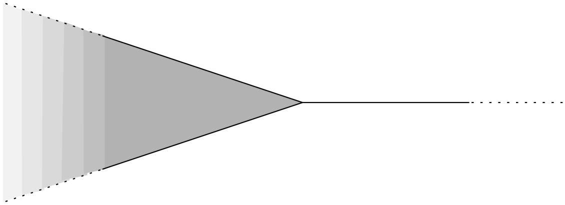
Definition

A metric measure space $(X, \mathrm{d}, \mathfrak{m})$ satisfies the *measure contraction property*, $MCP(0, N)$ if for every $x_0 \in X$ and $A \subset X$ with $0 < \mathfrak{m}(A) < \infty$ there exists a geodesic $(\mu_t)_{t \in [0,1]}$ in $\mathscr{P}(X)$ such that $\mu_0 = \delta_{x_0}$, $\mu_1 = (\mathfrak{m}(A))^{-1}\mathfrak{m}|_A$ and for $t \in [0, 1]$,

$$\frac{\mathrm{d}\mu_t}{\mathrm{d}\mathfrak{m}} \leq \frac{1}{t^N \mathfrak{m}(A)}.$$

Spaces with Ricci curvature lower bounds

Consider the following metric measure space: Projection of the measure to the x_1-axis is Lebesgue, distance induced by $|\cdot|_\infty$.



- It satisfies $MCP(0,3)$. (Ketterer-R.)
- It satisfies a local Poincaré inequality, hence is a Lipschitz differentiability space.
- Parts of different dimensions (top., Assouad, Nagata, charts).
- Still unique tangents, and hence subFinsler Carnot groups almost everywhere ($\mathbb{R}$ and $(\mathbb{R}^2, |\cdot|_\infty)$).

Uniqueness of measures

Suppose we have a metric measure space $(X, \mathrm{d}, \mathfrak{m})$ with some nice property. If $(X, \mathrm{d}, \mathfrak{m}')$ has the same property, how are $\mathfrak{m}$ and $\mathfrak{m}'$ related? For example, are they absolutely continuous w.r.t. each other?

- Doubling measures: can be singular.
- Lipschitz differentiability space: can be singular (Schioppa)
- *MCP*-space?

Notice that in $\mathbb{R}^n$ a doubling measure satisfying the Poincaré inequality is absolutely continuous w.r.t. the Lebesgue measure. (Csörnyei-Jones)

A measure satistying a weaker qualitative variant of *MCP* on $\mathbb{R}^n$ is absolutely continuous w.r.t. the Lebesgue measure. (Cavalletti-Mondino)

Thanks!