Free Functional Inequalities on the Circle

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Classical Case

M a Riemanian manifold,

$$W_2^2(\mu,\nu) := \inf_{\pi \in \Pi(\mu,\nu)} \iint d(x,y)^2 \pi(dx,dy),$$

$\Pi(\mu,\nu)$: probability on $M \times M$ with marginals μ and ν.

1. This is a metric for the weak convergence on probabilities with finite second moment.
2. There is a unique map $T(x) = \exp_x(-\nabla f(x))$, f is $d^2/2$-concave such that $T_{\#}\nu = \mu$ and $\pi = (T, Id)_{\#}\nu$ is the optimal plan:

$$W_2^2(\mu,\nu) = \int d(x,T(x))^2 \nu(dx).$$

Talagrand, Log-Sobolev and HWI

1. A measure ν is said to satisfy $T(\rho)$ if for any probability μ

$$\rho W_2^2(\mu,\nu) \leq H(\mu|\nu).$$

$H(\mu|\nu) = \int \frac{d\mu}{d\nu} \log(\frac{d\mu}{d\nu}) d\nu.$

2. ν satisfies $LSI(\rho)$ if for any μ,

$$H(\mu|\nu) \leq \frac{1}{4\rho} I(\mu|\nu)$$

where $I(\mu|\nu) = \int \left|\nabla \log\left(\frac{d\mu}{d\nu}\right)\right|^2 d\mu = 4 \int \left|\nabla \sqrt{\frac{d\mu}{d\nu}}\right|^2 d\nu.$

3. ν satisfies $HWI(\rho)$ inequality if for any μ:

$$H(\mu|\nu) \leq W_2(\mu,\nu) \sqrt{I(\mu|\nu)} - \rho W_2^2(\mu,\nu).$$

4. ν satisfies $P(\rho)$ if for any smooth f

$$\mathrm{Var}_\nu(f) \leq \int |\nabla f|^2 d\nu.$$

Theorem (Otto and Villani)

Assume $\nu = e^{-\xi(x)}dx$ is a probability measure on M.

1. *If $\mathrm{Hess}\xi + \mathrm{Ric} \geq 2\rho$ for some $\rho \in \mathbb{R}$, then HWI(ρ) holds true.*
2. *LSI(ρ) implies $T(\rho)$.*
3. *If $\mathrm{Hess}\xi + \mathrm{Ric} \geq 2C$ is convex for a certain $C \in \mathbb{R}$, then $T(\rho)$ $(\rho > 0)$ implies LSI(ρ'), $\rho' > 0$.*
4. *$T(\rho)$, LSI(ρ) and HWI(ρ) for $\rho > 0$ imply $P(\rho)$.*

Random matrices and the entropy change

$\mathbb{P}_n^Q(dM) = \frac{1}{Z_n^Q} e^{-nTr\, Q(M)} dM$ on Hermitian matrices.

1. $\frac{1}{n^2} H(\mathbb{P}_n^{\tilde{Q}} | \mathbb{P}_n^Q) =$
$\frac{1}{n^2} \log Z_n^{\tilde{Q}} - \frac{1}{n^2} \log Z_n^Q + \frac{1}{n} \int Tr(\tilde{Q}(M) - Q(M)) \mathbb{P}_n^{\tilde{Q}}(dM)$
2. For $\eta_n(x) = \frac{1}{n} \sum_{k=1}^n \delta_{x_i}$

$$Z_n^Q = \int e^{-n \sum_{i=1}^n Q(x_i)} \prod_{1 \le i < j \le n} (x_i - x_j)^2 \prod_{i=1}^n dx_i$$

$$= \int \exp\left(-n^2 \left(\frac{1}{n} \sum_{i=1}^n Q(x_i) - \frac{1}{n^2} \sum_{1 \le i \ne j \le n} \log |x_i - x_j| \right) \right) \prod_{i=1}^n dx_i$$

$$= \int e^{-n^2 \left(\int Q(t) \eta_n(x)(dt) - \iint_{t \ne s} \log |t - s| \eta_n(x)(dt) \eta_n(x)(ds) \right)} \prod_{i=1}^n dx_i$$

3. $E_Q(\mu) = \int Q d\mu - \iint \log |x - y| \mu(dx) \mu(dy)$ and $\eta_n \to \mu_Q$

$$\frac{1}{n^2} H(\mathbb{P}_n^{\tilde{Q}} | \mathbb{P}_n^Q) \to E_{\tilde{Q}}(\mu_{\tilde{Q}}) - E_Q(\mu_Q) + \int (\tilde{Q} - Q) d\mu_{\tilde{Q}}$$

Random matrices and the limit of the transportation

If $Q'' \geq \rho$, then

$$\rho W_2^2(\mathbb{P}_n^{\tilde{Q}}, \mathbb{P}_n^{Q})/n^2 \leq H(\mathbb{P}_n^{\tilde{Q}} | \mathbb{P}_n^{Q})/n^2$$

and in the limit,

$$\rho W_2^2(\mu_{\tilde{Q}}, \mu_Q) \leq E_{\tilde{Q}}(\mu_{\tilde{Q}}) - E_Q(\mu_Q) + \int (\tilde{Q} - Q) d\mu_{\tilde{Q}}$$

Free Entropy

Q smooth on a closed subset K of $\mathbb{C}$ such that $\lim_{|x|\to\infty} \frac{Q(x)}{\log(1+|x|^2)} = \infty$ if K is unbounded.
Cases we consider $K = \mathbb{R}$, S the unit circle or $\mathbb{C}$.

$$E_Q(\mu) = \int Q(x)\mu(x) - \iint \log|x-y|\mu(dx)\mu(dy).$$

There is a unique probability measure μ_Q such that

$$E_Q := E_Q(\mu_Q) = \inf_{\mu\in\mathcal{P}(K)} E_Q(\mu).$$

In addition, μ_Q has compact support.

Variational Characterization

The variational characterization of μ_Q:

$$Q(x) \geq 2 \int \log |x - y| \mu_Q(dx) + C \quad \text{with equality for } x \in supp(\mu).$$

If $K = \mathbb{R}$ and the support of μ is a union of intervals, then for a.e. $x \in supp(\mu)$:

$$Q'(x) = \int \frac{2}{x - y} \mu_Q(dx).$$

For the circle case $S = [-\pi, \pi)$, a.s. on the support of μ_Q

$$Q'(x) = \int \cot\left(\frac{x - y}{2}\right) \mu_Q(dx).$$

The relative free entropy is defined as

$$E_Q(\mu|\mu_Q) = E_Q(\mu) - E_Q(\mu_Q).$$

It is always positive, unless $\mu = \mu_Q$.

If $Q(x) = x^2/2$ on $\mathbb{R}$, then the minimizer of the free entropy μ_Q is given by the semicircular law

$$\mu_Q(dx) = \frac{1}{2\pi}\mathbb{1}_{[-2,2]}(x)\sqrt{4-x^2}dx.$$

with $E_Q = 3/4$.
If $Q = 0$ on S, then the equilibrium measure $\mu_Q = \alpha$ is the Haar measure on S.

The Free Information

$$I_Q(\mu|\mu_Q) = \int (H\mu(x) - Q'(x))^2 \mu(dx)$$

where

$$H\mu(x) = \begin{cases} \int \frac{2}{x-y} \mu(dy) & K = \mathbb{R} \\ \int \cot\left(\frac{x-y}{2}\right) \mu(dy) & K = S = [-\pi, \pi) \end{cases}$$

in the principal value sense.

Free Talagrand and Log-Sobolev on $\mathbb{R}$

Theorem (Biane & Voiculescu)

If $Q(x) = x^2/2$ *then*

$$\frac{1}{2}W_2^2(\mu, \mu_Q) \leq E_Q(\mu|\mu_Q).$$

Theorem (Hiai & Ueda & Petz)

If $Q''(x) \geq \rho$, *then*

$$\rho W_2^2(\mu, \mu_Q) \leq E_Q(\mu|\mu_Q).$$

Theorem (Biane & Speicher)

If $Q''(x) \geq \rho$ *is convex for* $\rho > 0$, *then*

$$E(\mu, \mu_Q) \leq \frac{1}{4\rho} I_Q(\mu|\mu_Q).$$

Unitary random matrix limit

$\mathbb{P}_n(dM) = \frac{1}{Z_n^Q} e^{-nTrQ(M)}\mathcal{H}(dM)$ on $n \times n$ unitary matrices, where $\mathcal{H}$ is the Haar measure on the unitary group, $U(n)$.
$Ric \geq n(n-1)/2$ except in one direction where it is 0.
The fix is to consider the subgroup G_n of matrices of determinant 1 and on this $Ricci = n(n-1)/2$.
Random matrix approximations gives

Theorem (Hia-Ueda-Petz)

If $Q'' \geq \rho - 1/2$, then the transportation inequality

$$\rho W_2^2(\mu, \mu_Q) \leq E(\mu|\mu_Q).$$

where W_2 is the standard Wasserstein distance.
$LSI(\rho)$ holds in the form

$$4\rho E(\mu|\mu_Q) \leq \int (H\mu - Q')^2 d\mu - \left(\int Q' d\mu\right)^2 := I(\mu|\mu_Q).$$

Using random matrices one can get the free transportation, *LS* but not *HWI*.
There is also a free Poincaré inequality which can be deduced from the matrix models in some particular cases of the potential Q. Apply the classical Poincare to $F(M) = Tr\, f(M)$:

$$2\rho\, Var_{\mathbb{P}_n}(F) \leq \int |F'|^2 d\mathbb{P}_n$$

which becomes

$$2\rho \iint \left|\frac{f(z) - f(w)}{z - w}\right|^2 \alpha(dz)\,\alpha(dw) \leq \int |f'|^2 d\mu_Q - \left|\int f' d\mu_Q\right|^2 .$$

Free Poincaré on the Circle

Definition

μ on S satisfies a free Poincaré inequality, $P(\rho)$, $\rho > 0$ if

$$2\rho \iint \left|\frac{f(z)-f(w)}{z-w}\right|^2 \alpha(dz)\,\alpha(dw) \leq \int |f'|^2 d\mu - \left|\int f'\,d\mu\right|^2$$

for any smooth $f : S \to \mathbb{C}$ where α is the Haar measure.

If $\mu = \alpha$, and

$$(\mathcal{N}\phi)(z) = \int \frac{(\phi'(z)-\phi'(w))(z+w)}{i(z-w)}\alpha(dw)$$

$$(\mathcal{E}\phi)(z) = -2\int \log|z-w|\phi(w)\alpha(dw)$$

then $\mathcal{N}z^{\pm n} = nz^{\pm n}$, $\mathcal{E}z^{\pm n} = \frac{1}{n}z^{\pm n}$ $(n \neq 0)$ and $\mathcal{N}^2 f = \mathcal{L}f = -f''$.

$$P(1/2) \iff \langle \mathcal{N}f, f\rangle \leq \langle \mathcal{L}f, f\rangle.$$

If μ has $P(\rho)$, then necessarily its support is the whole S. Moreover, if μ has a density w w.r.t. α, then w must be positive on S.

Theorem

If $Q'' \geq \rho - 1/2$, then

$$2\rho \iint \left|\frac{f(z) - f(w)}{z - w}\right|^2 \alpha(dz)\,\alpha(dw) \leq \int |f'|^2 d\mu_Q - \left|\int f' d\mu_Q\right|^2.$$

$$\langle \mathcal{N}\phi, \phi\rangle = \langle \mathcal{E}(\phi' - \int \phi' d\mu_Q), \phi' - \int \phi' d\mu_Q\rangle.$$

$$\langle \mathcal{E}(\phi' - \int \phi' d\mu_Q), \phi' - \int \phi' d\mu_Q\rangle \leq \langle \phi' - \int \phi' d\mu_Q, \phi' - \int \phi' d\mu_Q\rangle$$

$$= \left\langle \frac{1}{1 - \mathcal{N}Q}\left(\phi' - \int \phi' d\mu_Q\right), \phi' - \int \phi' d\mu_Q \right\rangle_{\mu_Q}$$

$$= \int \frac{\left|\phi' - \int \phi'\, d\mu_Q\right|^2}{1 - \mathcal{N}Q} d\mu_Q \leq \frac{1}{2\rho} \int \left|\phi' - \int \phi'\, d\mu_Q\right|^2 d\mu_Q,$$

A refinement in the case of the uniform measure

$$\partial_c(z^n) = \begin{cases} 0 & n = 0 \\ \sum_{i=1}^{n-1} z^i \otimes z^{n-i} & n \geq 1 \\ \sum_{i=1}^{n-1} z^{-i} \otimes z^{n+i} & n \leq -1 \end{cases}$$

Theorem

For any $k \geq 1$,

$$\sum_{l=1}^{2k} \frac{(-1)^{l-1}}{l} \|\partial_c^{(l-1)} \phi'\|^2_{\alpha^{\otimes l}} \leq \langle \mathcal{N}\phi, \phi \rangle \leq \sum_{l=1}^{2k-1} \frac{(-1)^{l-1}}{l} \|\partial_c^{(l-1)} \phi'\|^2_{\alpha^{\otimes l}}.$$

The modified Wasserstein

For a measure μ on S,

$$\bar{\mu}(A) = \sum_{n \in \mathbb{Z}} \mu(\exp(A \cap [2n\pi, 2(n+1)\pi)).$$

$\bar{\mu}_u$ is simply the restriction of $\bar{\mu}$ to the interval $[u, u + 2\pi)$.

Definition

For $\mu, \nu \in \mathcal{P}(S)$,

$$\mathcal{W}_2(\mu, \nu) = \sup_{u,v \in [0,2\pi]} \left\{ W_2(\bar{\mu}_u, \bar{\nu}_v) : \int x \, \bar{\mu}_u(dx) = \int x \, \bar{\nu}_v(dx), \right\},$$

where $W_2(\bar{\mu}_u, \bar{\nu}_v)$ is the standard Wasserstein distance on the real line.

Proposition

1. $W_2(\mu,\nu) \le \mathcal{W}_2(\mu,\nu)$.
2. $\mathcal{W}_2(\delta_a,\delta_b) = \infty$ *unless* $a = b$.
3. $\mathcal{W}_2(\delta_a,\alpha) = \left(\frac{2\pi^3}{3}\right)^{1/3}$ *so* $\mathcal{W}_2(\delta_a,\delta_b) > \mathcal{W}_2(\delta_a,\alpha) + \mathcal{W}_2(\alpha,\delta_b)$.
4. *If* $\mu,\nu \in \mathcal{P}_{as}(S)$, *there exists* $t \in [0,2\pi]$ *s.t.* $\int x\,\bar{\mu}_t(dx) = \int x\,\bar{\nu}_t(dx)$. *Moreover, given* $u \in [0,2\pi]$, *there exists* $v \in [0,2\pi]$ *such that* $\int x\,\bar{\mu}_u(dx) = \int x\,\bar{\nu}_v(dx)$.
5. *If* $\mu,\zeta,\nu \in \mathcal{P}_{as}(S)$, *then* $\mathcal{W}_2(\mu,\nu) \le \mathcal{W}_2(\mu,\zeta) + \mathcal{W}_2(\zeta,\nu)$.
6. *On* $\mathcal{P}_{as}(S)$, *the topology induced by* $\mathcal{W}_2$ *is the topology of weak convergence.*
7. $\mu,\nu \in \mathcal{P}_{as}(S)$, *then* $\mathcal{W}_2^2(\mu,\nu) =$

$$\sup_{\substack{(u,v)\\(\lambda,f,g)}} \left\{ \int f\,d\bar{\mu}_u + \int g\,d\bar{\nu}_v : \begin{array}{l} \int x\,\bar{\mu}_u(dx) = \int x\,\bar{\nu}_v(dx) \\ f(x) + g(y) \le (x-y)^2 + \lambda(x-y),\ \forall x,y \in \mathbb{R} \end{array} \right.$$

Theorem

If $Q'' \geq \rho - 1/2$, then

$$\frac{\rho}{2}\mathcal{W}_2^2(\mu, \mu_Q) \leq E_Q(\mu|\mu_Q).$$

Even for $Q = 0$ ($\mu_Q = \alpha$) this is not sharp! I conjecture that in this case

$$\frac{1}{2}\mathcal{W}_2^2(\mu, \alpha) \leq E_Q(\mu|\alpha).$$

For $Q = 0$ this is equivalent to

$$\int (\theta(x) - x)^2 d\alpha_u(x) \leq \frac{1}{2}\iint ((\theta(x) - \theta(y) - (x - y)) \cot(\frac{x - y}{2})\alpha_u(dx)\alpha_u(d$$

$$- \iint \log\left(\frac{\sin((\theta(x) - \theta(y))/2)}{\sin((x - y)/2)}\right)\alpha_u(dx)\alpha_u(dy)$$

Because $\int (\theta(x) - x) d\alpha_u(x) = 0$,

$$\int (\theta(x) - x)^2 \alpha_u(dx) = \frac{1}{2}\iint ((\theta(x) - \theta(y) - (x - y))^2 \alpha_u(dx)\alpha_u(dy)$$

$$\int(\theta(x)-x)^2 d\alpha_u(x) \leq \frac{1}{2}\iint((\theta(x)-\theta(y)-(x-y))\cot(\frac{x-y}{2})\alpha_u(dx)\alpha_u(d
$$

$$- \iint \log\left(\frac{\sin((\theta(x)-\theta(y))/2)}{\sin((x-y)/2)}\right)\alpha_u(dx)\alpha_u(dy)$$

Because $\int(\theta(x)-x)d\alpha_u(x) = 0$,

$$\int(\theta(x)-x)^2\alpha_u(dx) = \frac{1}{2}\iint((\theta(x)-\theta(y)-(x-y))^2\alpha_u(dx)\alpha_u(dy)$$

and the rest follows from convexity of $\frac{1}{2}x^2 + \log(\sin(x))$ on $[0,\pi]$

$$\frac{1}{2}(a-b)^2 \leq (a-b)\cot(b) - \log\left(\frac{\sin(a)}{\sin(b)}\right)$$

with $a = (\theta(x)-\theta(y))/2$, $b = (x-y)/2$, $x > y$

Log-Sobolev can be proved based on the same strategy.
Also the following HWI holds

Theorem

$$E(\mu|\mu_Q) \leq \mathcal{W}_2(\mu, \mu_Q) I_Q(\mu)^{1/2} - \rho \mathcal{W}_2(\mu, \mu_Q)^2$$

with

$$I(\mu|\mu_Q) = \int (H\mu - Q')^2 d\mu - \left(\int Q' d\mu \right)^2$$

Theorem

If μ_Q is the equilibrium measure of E_Q for a C^3 potential Q such that $\frac{d\mu_Q}{d\alpha} > 0$ on S, then $LSI(\rho), T(\rho), HWI(\rho)$ imply $P(\rho)$ for μ_Q.

$$\mathcal{W}_2^2(\mu, \mu_Q) = \sup_{\lambda} \int f d\mu - \int g d\mu_Q : f(x) \leq g(y) + (x-y)^2 + \lambda(x-y)$$

$$= \sup_{\lambda, g} \{ \int U_1 g_\lambda d\mu - \frac{\lambda^2}{4} - \int g d\mu_Q \}$$

with $g_\lambda(x) = g(x + \lambda/2)$ and $(U_t g)(x) = \inf\{g(y) + \frac{(x-y)^2}{t}\}$. $T(\rho)$ implies

$$\rho \left(\int U_1 g_\lambda d\mu - \frac{\lambda^2}{4} - \int g d\mu_Q \right) \leq E_Q(\mu) - E_Q$$

or

$$0 \leq E_{Q - \rho U_1 g_\lambda} + \rho \frac{\lambda^2}{4} - E_Q + \rho \int g d\mu_Q$$

$$0 \leq E_{Q-\rho U_1 g_\lambda} + \rho\frac{\lambda^2}{4} - E_Q + \rho \int g d\mu_Q$$

Replacing g by tg and λ by $t\lambda$,
$(U_1(tg) = tU_t g = tg - t^2 g^2/4 + o(t^2))$

$$0 \leq -\frac{t^2}{2}\rho\langle \mathcal{N} g, g\rangle + \frac{t^2}{4}\int (g')^2 d\mu_Q - \frac{t^2}{2}\lambda \int g' d\mu_Q + t^2\frac{\lambda^2}{4} + o(t^2)$$

thus

$$2\rho\langle \mathcal{N} g, g\rangle \leq \int (g')^2 d\mu_Q - 2\lambda \int g' d\mu_Q + \lambda^2$$

for any λ. Minimizing in λ yields

$$2\rho\langle \mathcal{N} g, g\rangle \leq \int (g')^2 d\mu_Q - \left(\int g' d\mu_Q\right)^2$$

which is Poincaré.

Back to the classical case

Assume M is a Riemannian manifold and G a Lie Group acting isometrically on M. If $\mu(dx) = e^{-\xi(x)}dx$ with $Hess\xi + Ric \geq \rho$, then

$$Var_\mu(f) \leq \int |\nabla f|^2 d\mu - \left| \int D_G f d\mu \right|^2$$

In the case μ is invariant with respect to the action, the last integral is 0.
An interesting challenge is to see a similar modification for the transportation.

Thank You!