Free Functional Inequalities on the Circle

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Classical Case

\( M \) a Riemannian manifold,

\[
W_2^2(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \iint d(x, y)^2 \pi(dx, dy),
\]

\( \Pi(\mu, \nu) \) : probability on \( M \times M \) with marginals \( \mu \) and \( \nu \).

1. This is a metric for the weak convergence on probabilities with finite second moment.

2. There is a unique map \( T(x) = \exp_x(-\nabla f(x)) \), \( f \) is \( d^2/2 \)-concave such that \( T_\# \nu = \mu \) and \( \pi = (T, Id)_\# \nu \) is the optimal plan:

\[
W_2^2(\mu, \nu) = \int d(x, T(x))^2 \nu(dx).
\]
A measure $\nu$ is said to satisfy $T(\rho)$ if for any probability $\mu$

$$\rho \mathcal{W}_2^2(\mu, \nu) \leq H(\mu|\nu).$$

$$H(\mu|\nu) = \int \frac{d\mu}{dv} \log \left(\frac{d\mu}{dv}\right) dv.$$

$\nu$ satisfies $LSI(\rho)$ if for any $\mu$,

$$H(\mu|\nu) \leq \frac{1}{4\rho} I(\mu|\nu)$$

where $I(\mu|\nu) = \int \left| \nabla \log \left(\frac{d\mu}{dv}\right) \right|^2 d\mu = 4 \int \left| \nabla \sqrt{\frac{d\mu}{dv}} \right|^2 dv.$

$\nu$ satisfies $HWI(\rho)$ inequality if for any $\mu$:

$$H(\mu|\nu) \leq \mathcal{W}_2(\mu, \nu) \sqrt{I(\mu|\nu)} - \rho \mathcal{W}_2^2(\mu, \nu).$$

$\nu$ satisfies $P(\rho)$ if for any smooth $f$

$$\text{Var}_\nu(f) \leq \int |\nabla f|^2 dv.$$
Theorem (Otto and Villani)

Assume $\nu = e^{-\xi(x)} \, dx$ is a probability measure on $M$.

1. If $\text{Hess} \xi + \text{Ric} \geq 2\rho$ for some $\rho \in \mathbb{R}$, then $\text{HWI}(\rho)$ holds true.

2. $\text{LSI}(\rho)$ implies $T(\rho)$.

3. If $\text{Hess} \xi + \text{Ric} \geq 2C$ is convex for a certain $C \in \mathbb{R}$, then $T(\rho)$ ($\rho > 0$) implies $\text{LSI}(\rho')$, $\rho' > 0$.

4. $T(\rho)$, $\text{LSI}(\rho)$ and $\text{HWI}(\rho)$ for $\rho > 0$ imply $P(\rho)$. 
Random matrices and the entropy change

\[ \mathbb{P}_n^Q(dM) = \frac{1}{Z_n^Q} e^{-n \text{Tr} Q(M)} dM \] on Hermitian matrices.

1. \[ \frac{1}{n^2} H(\mathbb{P}_n^{\tilde{Q}} \| \mathbb{P}_n^Q) = \frac{1}{n^2} \log Z_n^{\tilde{Q}} - \frac{1}{n^2} \log Z_n^Q + \frac{1}{n} \int \text{Tr}(\tilde{Q}(M) - Q(M)) \mathbb{P}_n^{\tilde{Q}}(dM) \]

2. For \( \eta_n(x) = \frac{1}{n} \sum_{k=1}^n \delta_{x_i} \)

\[ Z_n^Q = \int e^{-n \sum_{i=1}^n Q(x_i)} \prod_{1 \leq i < j \leq n} (x_i - x_j)^2 \prod_{i=1}^n dx_i \]

\[ = \int \exp \left( -n^2 \left( \frac{1}{n} \sum_{i=1}^n Q(x_i) - \frac{1}{n^2} \sum_{1 \leq i \neq j \leq n} \log |x_i - x_j| \right) \right) \prod_{i=1}^n dx_i \]

\[ = \int e^{-n^2 \left( \int Q(t) \eta_n(x)(dt) - \int_{t\neq s} \log |t-s| \eta_n(x)(dt) \eta_n(x)(ds) \right)} \prod_{i=1}^n dx_i \]

3. \[ E_Q(\mu) = \int Q d\mu - \iint \log |x - y| \mu(dx) \mu(dy) \] and \( \eta_n \to \mu_Q \)

\[ \frac{1}{n^2} H(\mathbb{P}_n^{\tilde{Q}} \| \mathbb{P}_n^Q) \to E_{\tilde{Q}}(\mu_{\tilde{Q}}) - E_Q(\mu_Q) + \int (\tilde{Q} - Q) d\mu_{\tilde{Q}} \]
If $Q'' \geq \rho$, then

$$\rho W_2^2(\mathbb{P}_{\tilde{Q}n}, \mathbb{P}_Q^n)/n^2 \leq H(\mathbb{P}_{\tilde{Q}n}||\mathbb{P}_Q^n)/n^2$$

and in the limit,

$$\rho W_2^2(\mu_{\tilde{Q}}, \mu_Q) \leq E_{\tilde{Q}}(\mu_{\tilde{Q}}) - E_Q(\mu_Q) + \int (\tilde{Q} - Q) d\mu_{\tilde{Q}}$$
$Q$ smooth on a closed subset $K$ of $\mathbb{C}$ such that 
\[
\lim_{|x| \to \infty} \frac{Q(x)}{\log(1+|x|^2)} = \infty \text{ if } K \text{ is unbounded.}
\]
Cases we consider $K = \mathbb{R}$, $S$ the unit circle or $\mathbb{C}$.

\[
E_Q(\mu) = \int Q(x)\mu(x) - \iint \log|x - y|\mu(dx)\mu(dy).
\]

There is a unique probability measure $\mu_Q$ such that

\[
E_Q := E_Q(\mu_Q) = \inf_{\mu \in \mathcal{P}(K)} E_Q(\mu).
\]

In addition, $\mu_Q$ has compact support.
The variational characterization of $\mu_Q$:

$$Q(x) \geq 2 \int \log |x - y| \mu_Q(dx) + C \quad \text{with equality for } x \in \text{supp}(\mu).$$

If $K = \mathbb{R}$ and the support of $\mu$ is a union of intervals, then for a.e. $x \in \text{supp}(\mu)$:

$$Q'(x) = \int \frac{2}{x - y} \mu_Q(dx).$$

For the circle case $S = [-\pi, \pi)$, a.s. on the support of $\mu_Q$

$$Q'(x) = \int \cot \left( \frac{x - y}{2} \right) \mu_Q(dx).$$
The relative free entropy is defined as

\[ E_Q(\mu | \mu_Q) = E_Q(\mu) - E_Q(\mu_Q). \]

It is always positive, unless \( \mu = \mu_Q \).

If \( Q(x) = x^2/2 \) on \( \mathbb{R} \), then the minimizer of the free entropy \( \mu_Q \) is given by the semicircular law

\[ \mu_Q(dx) = \frac{1}{2\pi} \mathbb{1}_{[-2,2]}(x) \sqrt{4 - x^2} \, dx. \]

with \( E_Q = 3/4 \).

If \( Q = 0 \) on \( S \), then the equilibrium measure \( \mu_Q = \alpha \) is the Haar measure on \( S \).
\[ I_Q(\mu|\mu_Q) = \int (H\mu(x) - Q'(x))^2 \mu(dx) \]

where

\[ H\mu(x) = \begin{cases} \int \frac{2}{x-y} \mu(dy) & K = \mathbb{R} \\ \int \cot \left( \frac{x-y}{2} \right) \mu(dy) & K = S = [-\pi, \pi) \end{cases} \]

in the principal value sense.
Theorem (Biane & Voiculescu)

If \( Q(x) = x^2 / 2 \) then

\[
\frac{1}{2} W_2^2(\mu, \mu_Q) \leq E_Q(\mu|\mu_Q).
\]

Theorem (Hiai & Ueda & Petz)

If \( Q''(x) \geq \rho \), then

\[
\rho W_2^2(\mu, \mu_Q) \leq E_Q(\mu|\mu_Q).
\]

Theorem (Biane & Speicher)

If \( Q''(x) \geq \rho \) is convex for \( \rho > 0 \), then

\[
E(\mu, \mu_Q) \leq \frac{1}{4\rho} I_Q(\mu|\mu_Q).
\]
Unitary random matrix limit

\[ IP_n(dM) = \frac{1}{Z^n} e^{-n\text{Tr}Q(M)} \mathcal{H}(dM) \] on \( n \times n \) unitary matrices, where \( \mathcal{H} \) is the Haar measure on the unitary group, \( U(n) \).

\( \text{Ric} \geq n(n - 1)/2 \) except in one direction where it is 0. The fix is to consider the subgroup \( G_n \) of matrices of determinant 1 and on this \( \text{Ricci} = n(n - 1)/2 \).

Random matrix approximations gives

**Theorem (Hia-Ueda-Petz)**

If \( Q'' \geq \rho - 1/2 \), then the transportation inequality

\[ \rho W_2^2(\mu, \mu_Q) \leq E(\mu|\mu_Q). \]

where \( W_2 \) is the standard Wasserstein distance.

\( LSI(\rho) \) holds in the form

\[ 4\rho E(\mu|\mu_Q) \leq \int (H\mu - Q')^2 d\mu - \left( \int Q' d\mu \right)^2 := I(\mu|\mu_Q). \]
Using random matrices one can get the free transportation, $LS$ but not $HWI$.

There is also a free Poincaré inequality which can be deduced from the matrix models in some particular cases of the potential $Q$. Apply the classical Poincare to $F(M) = \text{Tr } f(M)$:

$$2\rho \text{Var}_{\mathbb{P}_n}(F) \leq \int |F'|^2 d\mathbb{P}_n$$

which becomes

$$2\rho \int \int \left| \frac{f(z) - f(w)}{z - w} \right|^2 \alpha(dz) \alpha(dw) \leq \int |f'|^2 d\mu_Q - \left| \int f' d\mu_Q \right|^2.$$
**Free Poincaré on the Circle**

### Definition

μ on S satisfies a free Poincaré inequality, \( P(\rho) \), \( \rho > 0 \) if

\[
2\rho \iint \left| \frac{f(z) - f(w)}{z - w} \right|^2 \alpha(dz) \alpha(dw) \leq \int |f'|^2 d\mu - \left| \int f' d\mu \right|^2
\]

for any smooth \( f : S \to \mathbb{C} \) where \( \alpha \) is the Haar measure.

If \( \mu = \alpha \), and

\[(N\phi)(z) = \int \frac{(\phi'(z) - \phi'(w))(z + w)}{i(z - w)} \alpha(dw)\]

\[(E\phi)(z) = -2 \int \log |z - w| \phi(w) \alpha(dw)\]

then \( N z^{\pm n} = nz^{\pm n} \), \( E z^{\pm n} = \frac{1}{n} z^{\pm n} \) \((n \neq 0)\) and \( N^2 f = \mathcal{L}f = -f'' \).

\[P(1/2) \iff \langle N f, f \rangle \leq \langle \mathcal{L} f, f \rangle.\]
If $\mu$ has $P(\rho)$, then necessarily its support is the whole $S$. Moreover, if $\mu$ has a density $w$ w.r.t. $\alpha$, then $w$ must be positive on $S$.

**Theorem**

If $Q'' \geq \rho - 1/2$, then

$$2\rho \int \int \frac{|f(z) - f(w)|^2}{z-w} \alpha(dz) \alpha(dw) \leq \int |f'|^2 d\mu_Q - \left| \int f' d\mu_Q \right|^2.$$
A refinement in the case of the uniform measure

\[
\partial_c(z^n) = \begin{cases} 
0 & n = 0 \\
\sum_{i=1}^{n-1} z^i \otimes z^{n-i} & n \geq 0 \\
\sum_{i=1}^{n-1} z^{-i} \otimes z^{n+i} & n \leq -1
\end{cases}
\]

**Theorem**

*For any* \( k \geq 1 \),

\[
\sum_{l=1}^{2k} \frac{(-1)^{l-1}}{l} \|\partial_c^{(l-1)} \phi'\|_{\alpha^{\otimes l}}^2 \leq \langle \mathcal{N} \phi, \phi \rangle \leq \sum_{l=1}^{2k-1} \frac{(-1)^{l-1}}{l} \|\partial_c^{(l-1)} \phi'\|_{\alpha^{\otimes l}}^2.
\]
For a measure $\mu$ on $S$,

$$\bar{\mu}(A) = \sum_{n \in \mathbb{Z}} \mu(\exp(A \cap [2n\pi, 2(n+1)\pi))).$$

$\bar{\mu}_u$ is simply the restriction of $\bar{\mu}$ to the interval $[u, u + 2\pi)$.

**Definition**

For $\mu, \nu \in \mathcal{P}(S)$,

$$W_2(\mu, \nu) = \sup_{u, v \in [0, 2\pi]} \left\{ W_2(\bar{\mu}_u, \bar{\nu}_v) : \int x \bar{\mu}_u(dx) = \int x \bar{\nu}_v(dx), \right\},$$

where $W_2(\bar{\mu}_u, \bar{\nu}_v)$ is the standard Wasserstein distance on the real line.
Proposition

1. $W_2(\mu, \nu) \leq W_2(\mu, \nu)$.
2. $W_2(\delta_a, \delta_b) = \infty$ unless $a = b$.
3. $W_2(\delta_a, \alpha) = \left(\frac{2\pi^3}{3}\right)^{1/3}$ so
   $W_2(\delta_a, \delta_b) > W_2(\delta_a, \alpha) + W_2(\alpha, \delta_b)$.
4. If $\mu, \nu \in \mathcal{P}_{as}(S)$, there exists $t \in [0, 2\pi]$ s.t.
   $\int x \mu_t(dx) = \int x \nu_t(dx)$. Moreover, given $u \in [0, 2\pi]$, there
   exists $v \in [0, 2\pi]$ such that $\int x \mu_u(dx) = \int x \nu_v(dx)$.
5. If $\mu, \zeta, \nu \in \mathcal{P}_{as}(S)$, then $W_2(\mu, \nu) \leq W_2(\mu, \zeta) + W_2(\zeta, \nu)$.
6. On $\mathcal{P}_{as}(S)$, the topology induced by $W_2$ is the topology of
   weak convergence.
7. $\mu, \nu \in \mathcal{P}_{as}(S)$, then $W_2^2(\mu, \nu) =$

\[
\sup_{(u, v) \in (\lambda, f, g)} \left\{ \int f \, d\mu_u + \int g \, d\nu_v : \int x \mu_u(dx) = \int x \nu_v(dx) \right\}
\]
\[
f(x) + g(y) \leq (x - y)^2 + \lambda(x - y), \forall x, y \in \mathbb{R}
\]
Theorem

If $Q'' \geq \rho - 1/2$, then

$$\frac{\rho}{2} W_2^2(\mu, \mu_Q) \leq E_Q(\mu|\mu_Q).$$

Even for $Q = 0$ ($\mu_Q = \alpha$) this is not sharp! I conjecture that in this case

$$\frac{1}{2} W_2^2(\mu, \alpha) \leq E_Q(\mu|\alpha).$$

For $Q = 0$ this is equivalent to

$$\int (\theta(x) - x)^2 d\alpha_u(x) \leq \frac{1}{2} \iint ((\theta(x) - \theta(y) - (x - y)) \cot(\frac{x - y}{2}) \alpha_u(dx) \alpha_u(dy)$$

$$- \iint \log \left( \frac{\sin((\theta(x) - \theta(y))/2))}{\sin((x - y)/2)} \right) \alpha_u(dx) \alpha_u(dy)$$

Because $\int (\theta(x) - x) d\alpha_u(x) = 0$,

$$\int (\theta(x)-x)^2 \alpha_u(dx) = \frac{1}{2} \iint ((\theta(x)-\theta(y)-(x-y))^2 \alpha_u(dx) \alpha_u(dy)$$
\[ \int (\theta(x) - x)^2 \, d\alpha_u(x) \leq \frac{1}{2} \int \int ((\theta(x) - \theta(y) - (x - y)) \cot\left(\frac{x - y}{2}\right) \alpha_u(dx) \alpha_u(dy) \]

\[ \quad - \int \int \log\left(\frac{\sin((\theta(x) - \theta(y))/2)}{\sin((x - y)/2)}\right) \alpha_u(dx) \alpha_u(dy) \]

Because \( \int (\theta(x) - x) \, d\alpha_u(x) = 0, \)

\[ \int (\theta(x) - x)^2 \, d\alpha_u(dx) = \frac{1}{2} \int \int ((\theta(x) - \theta(y) - (x - y))^2 \, \alpha_u(dx) \alpha_u(dy) \]

and the rest follows from convexity of \( \frac{1}{2}x^2 + \log(\sin(x)) \) on \([0, \pi]\)

\[ \frac{1}{2}(a - b)^2 \leq (a - b) \cot(b) - \log\left(\frac{\sin(a)}{\sin(b)}\right) \]

with \( a = (\theta(x) - \theta(y))/2, \quad b = (x - y)/2, \quad x > y \)
Log-Sobolev can be proved based on the same strategy. Also the following HWI holds

**Theorem**

\[
E(\mu | \mu_Q) \leq \mathcal{W}_2(\mu, \mu_Q) I_Q(\mu)^{1/2} - \rho \mathcal{W}_2(\mu, \mu_Q)^2
\]

with

\[
I(\mu | \mu_Q) = \int (H\mu - Q')^2 d\mu - \left( \int Q' d\mu \right)^2
\]
Theorem

If $\mu_Q$ is the equilibrium measure of $E_Q$ for a $C^3$ potential $Q$ such that $\frac{d\mu_Q}{d\alpha} > 0$ on $S$, then $\text{LSI}(\rho), T(\rho), \text{HWI}(\rho)$ imply $P(\rho)$ for $\mu_Q$.

\[ W_2^2(\mu, \mu_Q) = \sup_{\lambda} \int f \, d\mu - \int g \, d\mu_Q : f(x) \leq g(y) + (x - y)^2 + \lambda(x - y) \]

\[ = \sup_{\lambda, g} \left\{ \int U_1 g_\lambda d\mu - \frac{\lambda^2}{4} - \int g \, d\mu_Q \right\} \]

with $g_\lambda(x) = g(x + \lambda/2)$ and $(U_t g)(x) = \inf\{g(y) + \frac{(x-y)^2}{t}\}$. $T(\rho)$ implies

\[ \rho \left( \int U_1 g_\lambda d\mu - \frac{\lambda^2}{4} - \int g \, d\mu_Q \right) \leq E_Q(\mu) - E_Q \]

or

\[ 0 \leq E_{Q-\rho U_1 g_\lambda} + \rho \frac{\lambda^2}{4} - E_Q + \rho \int g \, d\mu_Q \]
0 \leq E_Q - \rho u_1 g_\lambda + \rho \frac{\lambda^2}{4} - E_Q + \rho \int g d\mu_Q

Replacing \( g \) by \( tg \) and \( \lambda \) by \( t\lambda \),
(\( U_1(tg) = tU_t g = tg - t^2 g^2 / 4 + o(t^2) \))

\[
0 \leq -\frac{t^2}{2} \rho \langle \mathcal{N} g, g \rangle + \frac{t^2}{4} \int (g')^2 d\mu_Q - \frac{t^2}{2} \lambda \int g' d\mu_Q + t^2 \frac{\lambda^2}{4} + o(t^2)
\]

thus

\[
2\rho \langle \mathcal{N} g, g \rangle \leq \int (g')^2 d\mu_Q - 2\lambda \int g' d\mu_Q + \lambda^2
\]

for any \( \lambda \). Minimizing in \( \lambda \) yields

\[
2\rho \langle \mathcal{N} g, g \rangle \leq \int (g')^2 d\mu_Q - \left( \int g' d\mu_Q \right)^2
\]

which is Poincaré.
Assume $M$ is a Riemannian manifold and $G$ a Lie Group acting isometrically on $M$. If $\mu(dx) = e^{-\xi(x)}\,dx$ with $\text{Hess}\xi + \text{Ric} \geq \rho$, then

$$\text{Var}_\mu(f) \leq \int |\nabla f|^2 \,d\mu - \left| \int D_G f \,d\mu \right|^2$$

In the case $\mu$ is invariant with respect to the action, the last integral is 0. An interesting challenge is to see a similar modification for the transportation.
Thank You!