OPTIMAL CAPITAL GROWTH WITH SHORTFALL PENALTIES

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Abstract

In capital growth under uncertainty, an investor must determine how much capital to invest in riskless and risky instruments at each point in time, with a focus on the trajectory of accumulated capital to a planning horizon. In this paper the traditional capital growth model and modifications to control risk are developed. A mixture model based on Markov transitions between normally distributed market regimes is used for the dynamics of asset prices. Decisions on investment in assets are based on a constrained growth model, where the trajectory of wealth is required to exceed a specified path over time with high probability, and the path violations are penalized using a convex loss function.

1 Introduction

In dynamic stochastic systems, the key component is the process which changes the state of the system between time points. If the transition process can be controlled with decision variables, the state of the system at each time can be manipulated, and various parameter settings can be compared based on the associated state distributions. The distribution may be at a cross-section in time, i.e., at a planning horizon, or the distribution may be over trajectories across time. Whether it is from the perspective of the initial state and the terminal state, or the path from initial to terminal state, the primary feature of the process is the total change in state. That change can be represented by the rate of growth/decay.

In capital growth under uncertainty, an investor must determine how much capital to invest in riskless and risky instruments at each point in time, with a focus on the trajectory of accumulated capital to a planning horizon. Assuming prices are not affected by individual investments but rather aggregrate investments, individual decisions are made based on the projected price process given the history of prices to date. An investment strategy which has generated considerable interest is the growth optimal or Kelly strategy, where the expected logarithm of wealth is maximized. (Kelly, 1956.) Researchers such as Thorp (1971), Hausch, Ziemba and Rubinstein (1981), Grauer and Hakansson (1986, 1987), and Mulvey and Vladimirow (1992) have used the optimal growth strategy to compute portfolio weights in multi-asset and worldwide asset allocation problems. The wealth distribution of this strategy has many attractive characteristics (see, e.g. Breiman 1961; Hakansson, 1970, 1971; Markowitz, 1976, 2006; Roll, 1973; Hakansson and Ziemba, 1995; Rubinstein 1977; MacLean, Thorp and Ziemba, 2010; and Ziemba 2012).

The optimal growth strategy has many advantages, particularly in the long run. However, the literature points to a number of problems with such an aggressive strategy.

[1] The fallback in capital in the short to medium term may be unsustainable, i.e., capital could drop below an established operating minimum.

[2] The expectations (forecasts) for asset prices may be inaccurate. This could result from new market dynamics, or simply the volatility of markets. Errors in forecasts will translate into misdirected strategies which are suboptimal and inefficient. The downside risk from poor returns and the model risk from price estimation can be controlled, but the decision model needs to be modified.

The most common downside risk measure is Value at Risk (Jorion, 1997). VaR has been studied extensively (Artzner, et. Al. 1999, Gavaronski 2000). Basak and Shapiro (2001) consider VaR in a model with CRRA utility, where there is a probabilistic constraint imposed. Although VaR is an industry standard it has very serious weaknesses in controlling risk. The most serious shortcoming is the insensitivity to very large losses which have small probability - the essence of risk. Measures based on lower partial moments (incomplete means) such as CVaR (Rockefeller and Uryasev, 2000) and convex risk measures based on target violations (*Cariño* and Ziemba, 1998) attempt to deal with this problem. We follow the latter approach where penalties for non-performance are greater and greater as non-performance increases.

The practice with VaR and CVaR is to define risk constraints at a planning horizon T. However, the path to that horizon may be problematic. In this paper the traditional capital growth model and modifications to control risk are developed. A mixture model based on Markov transitions between market regimes is used for the dynamics of asset prices. Decisions on investment in assets are based on a constrained growth model, where the trajectory of wealth is required to exceed a specified path over time, and the path violations are penalized in a convex utility.

2 Market Structure

The wealth accumulation process is a stochastic dynamic system which depends on the allocation of capital to investment opportunities and the changing prices of those assets. A standard model for price dynamics is geometric Brownian motion. It is known that this model fails to capture important characteristics of asset prices, notably price distributions which are not log-normal and time dependent volatility. A flexible framework which accomodates observed price behavior is a Markov regime switching model, where the dynamics within a regime follow the standard geometric Brownian motion and the parameters in the dynamics vary by regime. Hamilton (1989) successfully applied the Markov model to US GDP data and charcterized the changing pattern of the US economy. Ang and Bekaert (2002) used regime shifts in a study of international asset allocation. Guidolin and Timmermann (2006) provided important insights into how investments vary across market regimes. The regimes make good economic sense, and the regime switching market structure is very amenable to analysis.

2.1 Switching Regime Model

Consider a competitive financial market with n assets whose prices are stochastic dynamic processes, and a single asset whose price is non-stochastic. Let the vector of prices at time t be

$$P(t) = (P_0(t), P_1(t), ..., P_n(t))', \tag{1}$$

where $P_0(t)$ is the price of the risk free asset, with rate of return r_t at time t. It will be assumed that the financial market is separated into m distinct regimes. Suppose the market is in regime k at time t, and let $Y_{ik}(t) = \ell n P_{ik}(t), i = 0, ..., n$ be the log-prices in regime $k, k = 1, ..., m$. The price dynamics within regime k are defined by the stochastic differential equations

$$dY_{0k}(t) = r_t dt \tag{2}$$

$$dY_k(t) = \alpha_k dt + \Delta_k dZ_k, k = 1, ..., m \tag{3}$$

where $Y_k(t) = \begin{pmatrix} Y_{1k}(t) \\ \vdots \\ Y_{nk}(t) \end{pmatrix}, \alpha_k = \begin{pmatrix} \alpha_{1k} \\ \vdots \\ \alpha_{nk} \end{pmatrix}, \Delta_k = (\delta_{ijk}), dZ_k = \begin{pmatrix} dZ_{1k} \\ \vdots \\ dZ_{nk} \end{pmatrix}$, where $dZ_{ik}, i = 1, ..., n$ are independent Brownian motions.

So the risky asset prices within a regime are assumed to have a joint log-normal distribution. The regimes over time $\{S_1, \ldots, S_t, ...\}$ follow a Markov chain. The probability of switching from regime i at time t to regime j at time $t + 1$ is $P[S_{t+1} = j | S_t =$

$i] = p_{ij}$. The transition between regimes is defined by the (stationary) transition matrix $P = \begin{pmatrix} p_{11} & \cdots & p_{1m} \\ \vdots & \vdots & \vdots \\ p_{m1} & \cdots & p_{mm} \end{pmatrix}$. Given an initial distribution over regimes $p(0) = (p_1(0), ..., p_m(0))$, the subsequent regime distributions $p(t)$ are determined by transitions. If the within regime parameters are $\theta_k = (\alpha_k, \Delta_k), k = 1, \ldots, m$, then the switching factor model is defined by the parameters $\Theta = (p(0), P, \theta_1, \ldots, \theta_m)$.

The market structure given by the switching factor model has advantages: (i) the ability to estimate the parameters in the model from observations on the asset returns and (ii) the ability to define analytically tractable investment models. The standard estimation procedure is the Expectation Maximization (EM) algorithm (Dempster et al, 1993). The investment model developed in subsequent sections considers that parameter values are known/estimated and focuses on investment strategies which control risk. The aspect of risk which is attributable to estimation error is not considered, but the positive results in the literature with the Markov switching model and the EM algorithm are the basis for the defined market structure.

2.2 Wealth Equations

In financial markets, assets are traded at points in time and the return on assets leads to the accumulation of capital for an investor. In the analysis of trading strategies, the following assumtions are made:

1. All assets have limited liability.
2. There are no transactions costs, taxes, or problems with indivisibility of assets.
3. Capital can be borrowed or lent at the risk free interest rate at any level.
4. Short sales of all assets is allowed.

An investment strategy is a vector process

$$\{(x_0(t), X(t)), t \geq 0\} = \{(x_0(t), x_1(t), ..., x_n(t)), t \geq 0\} \quad (4)$$

where $\sum_{i=0}^{n} x_i(t) = 1$ for any t, with $x_0(t)$ the investment in the risk-free asset.

The change in wealth from an investment decision $X(t)$ is determined by the price process . Consider that investment decisions are made at discrete points in time, $t = 1, ..., T$, and the price dynamics and regime are constant between time points. Let $\Delta_k^2 = \Delta_k' \Delta_k$ and $\phi_k = \alpha_k + \frac{1}{2}\Delta_k^2 e, k = 1, ..., m$. Then the instantaneous change in wealth if the market in period t is in regime k is given by

$$dW_k(t) = [X'(t)(\phi_k - re) + r]W_k(t)\,dt + W_k(t)[X'(t)\,\Delta_k dZ_{jk}]. \quad (5)$$

Consider that the wealth at the beginning of period t is w, the regime is k in period t , and the period length is subsumed into the parameters. So if the period is one day, the parameter ϕ_k is the vector of expected daily returns and Δ_k is the covariance matrix for daily returns. Then the conditional wealth at the end of period t if the regime is k, given the fixed investment strategy $X(t)$, is

$$W_k(t) = w \cdot exp\left\{[X'(t)(\phi_k - re) + r - \frac{1}{2}X'(t)\Delta_k^2 X(t)] + X(t)'\Delta_k Z_k\right\}, \quad (6)$$

where $Z_k' = (Z_{1k}, ..., Z_{nk}), Z_{ik} \sim N(0,1)$. Let

$$R_k(X(t)) = exp\left\{[X'(t)(\phi_k - re) + r - \frac{1}{2}X'(t)\Delta_k^2 X(t)] + X(t)'\Delta_k Z_k\right\} \quad (7)$$

be the return on the investment $X(t)$ in assets in period t. The rate of return in regime k is

$$ln(R_k(X(t))) = \left[X'(t)(\phi_k - re) + r - \frac{1}{2}X'(t)\Delta_k^2 X(t)\right] + X'(t)\Delta_k Z_k. \quad (8)$$

Let $f_k(v), k = 1, ..., m$ be the normal density of $ln(R_k(X(t))$, the log-return given the regime is k. Then $E(ln(R_k(X(t))) = \mu_k(t)$ and $\sigma(ln(R_k(X(t))) = \sigma_k(t)$, where $\mu_k(t) = X'(t)(\phi_k - re) + r - \frac{1}{2}\sigma_k^2(t)$ and $\sigma_k^2(t) = X'(t)\Delta_k^2 X(t)$. Assume that the distribution over regimes is $(p_1, ..., p_m)$ and let the unconditional return be $R(X(t))$. Then the unconditional distribution for log-returns is a mixture of normals $f(v) = p_1 f_1(v) + \ldots + p_m f_m(v)$.

For the stochastic process $R(X(t))$ a trajectory of the data process is associated with an outcome ω in the space Ω of all returns trajectories. The distributions over returns at each time t generate a probability measure P on Ω and the associated probability space (Ω, B, P). The sample space can be represented as $\Omega = \Omega_1 \times \cdots \times \Omega_T$, with $\omega_t \in \Omega_t$ the data at time t and $\Omega^t$the data up to and including time t. Subsets of Ω are of the form $A = A_1 \times \ldots \times A_T$. Let $R(\omega, X(t)), \omega \in \Omega$, be a returns trajectory, where $X(t)$ is an investment strategy which can depend on the data history but not on unknown future returns. The wealth trajectory is $W(\omega, t) = W(\omega, t-1)R(\omega, X(t))$.

3 Investment Model

Wealth is generated through investment in risky assets, but the trajectory of wealth can have large swings and the chance of falling below sustainable levels needs to be controlled. The characteristics of shortfalls (falling below benchmarks) are the rate/chance and the size. Both components are incorporated into our investment model, where shortfall rate is constrained at a specified level, and the shortfalls are penalized in the objective. The criterion in the objective is capital growth, which appears as a logarithmic utility. This is a risky objective, but the restrictions will play an important role in controlling downside risk.

3.1 Penalized Shortfall

We are concerned with trajectories which fall below a target path at discrete points in time. Consider a trajectory of the wealth process $W(t), 0 \leq t \leq T$, and the target/benchmark wealth path $w^*(t), 0 \leq t \leq T$, as shown in Figure 1. Two approaches to target paths are shown: (i) a growth path based on a desired rate, possibly at the risk free rate; (ii) a decay path based on a fallback rate. The targets are general and can vary from growth to decay over time.

If the trajectory is below the target at time t, $W(t) < w^*(t)$, then there is a penalty in the form of a wealth discount, $W(t)[1-\rho_t], \rho_t < 1$. Since the intention is to control large shortfalls, is is natural to make the penalty proportional to the shortfall, $\rho_t = \frac{w^*(t)-W(t)}{w^*(t)} = \frac{shortfall}{target}$. If $W(t) \geq w^*(t), \rho_t = 0$. Then

$$Discounted\ Wealth\colon\ W(t)\left[1-(1-\tfrac{W(t)}{w^*(t)})^+\right]^\gamma,$$

where penalty parameter γ captures the decision makers aversion to losses and the positive part is defined by $[y]^+ = y$ if $y > 0$ and $[y]^+ = 0$ if $y \leq 0$. This discounting approach works well with a logarithmic transformation since when $W(t) < w^*(t)$

$$ln\left(W(t)\left[1-(1-\frac{W(t)}{w^*(t)})^+\right]^\gamma\right) = ln\left(W(t)\right) - \gamma\left[ln(w^*(t) - ln(W(t))\right]^+ . \tag{9}$$

If $W(t) < w^*(t)$, the path shortfall is $w^*(t) - W(t)$ and the penalty $\gamma\left[ln(w^*(t) - ln(W(t))\right]$ is convex in the shortfall. The penalty parameter $\gamma \geq 1$ is a power factor.

The use of the logarithmitic transformation puts the focus on the growth rate of capital. That is, $ln(W(t)) = ln(W(t-1) + ln(R(t))$, where $ln(R(t))$ is the rate of return in period t. The optimal growth investment strategy has received a lot of attention (MacLean, Thorp and Ziemba, 2011). Although it has dominant returns in the long term, it carries significant short run risk. In the model proposed in this paper, the investors objective is to achieve capital growth with security, so that the chance and size of shortfalls in wealth is small. The shortfall can be considered from the perspective of the guaranteed return from the risk free rate, or a drawdown along the trajectory from initial wealth. The objective in optimal growth is formulated as the maximization of the logarithm of terminal wealth, and this decomposes into the period by period growth rates. If wealth is discounted as proposed, the same period by period decomposition applies.

To develop the wealth process and path shortfall, consider the return process $R(X(t))$. For the stochastic process $R(X(t))$ a trajectory of the data process is associated with an outcome ω in the space Ω of all returns trajectories. Let $R(\omega, X(t)), \omega \in \Omega$, be a returns trajectory, where $X(t)$ is an investment strategy which can depend on the data history but not on unknown future returns. The wealth trajectory is $W(\omega,t) = W(\omega,t-1)R(\omega,X(t))$. A requirement that the wealth trajectory lies above the path is $W(\omega,t) \geq w^*(t), t = 1,...,T$. For a set of trajectories $A \in B$, it could be required that all trajectories in the set satisfy the path condition: $W(\omega,t) \geq w^*(t), t = 1,...,T, \omega \in A$. Since $W(\omega,t) = w(t_0)\prod_{s=1}^{t} R(\omega,X(s)) = W(\omega,t-1)R(\omega,X(t))$, the additive logarithmitic process is $ln(W(\omega,t)) = ln(W(\omega,t-1)) + ln(R(\omega,X(t))$. The corresponding path condition is $ln(W(\omega,t)) \geq ln(w^*(t)), t = 1,...,T$.

If the path constraint is not satisfied, the model imposes a penalty at the point of violation. That is, the logarithm of discounted wealth at the horizon is $ln(w(t_0))+\sum_{t=1}^{T} ln(R(X(t))-\gamma\sum_{t=1}^{T}[ln(w^*(t)) - ln(W(t)]^+$. The penalty only applies if the path condition is violated.

3.2 Capital Growth with Security

If it is required that the path condition is satisfied $(1-\alpha)100\%$ of the time, the multiperiod capital growth problem, where the rate of shortfalls is controlled with a VaR constraint and the size is part of the objective, is written in probabilistic form as

$$max\left\{E\left[\sum_{t=1}^{T} ln(R(X(t)) - \gamma\sum_{t=1}^{T}[ln(w^*(t)) - ln(W(t-1)) - ln(R(X(t))]^+\right]\right\} \tag{10}$$

where

$$Pr[ln(R(X(t)) \geq ln(w^*(t)) - ln(W(t-1)), t = 1, \ldots, T] \geq 1-\alpha \tag{11}$$

$$X^\top(t)e = 1, t = 1, ..., T.$$

In the path VaR constraint it is required that the path condition is satisfied for a set of scenarios $A \in B$. If the measure of A is such that $P(A) \geq 1 - \alpha$, then the set is termed acceptable. For a given acceptance set $A = A_1 \times \ldots \times A_T, P(A) \geq 1 - \alpha$, a restricted form of the problem is

$$max \left\{ E \sum_{t=1}^{T} \left\{ ln(R(X(t)) - \gamma I_{\overline{A_t}}[ln(w^*(t)) - ln(W(t-1)) - ln(R(X(t)))] \right\} \right. \tag{12}$$

where

$$ln(R(\omega, X(t)) \geq ln(w^*(t)) - ln(W(\omega, t-1)), t = 1, \ldots, T, \omega \in A \tag{13}$$

$$X^\top(t)e = 1, t = 1, ..., T.$$

The acceptance set is the set of realizations (scenarios) for which the constraint is satisfied. This is a modification of the problem in MacLean et.al. (2004), with violations of the constraint penalized in the objective. The indicator $I_{\overline{A_t}}$ on the complement $\bar{A}_t$ captures the positive part, so it takes the value 1 if the constraint is violated and 0 otherwise.

It is assumed that there is an optimal solution to the problem and therefore there is an optimal acceptance set. That is, given the optimal acceptance set the solutions for the alternative formulations are the same. It was shown that the log transformation decomposed the final discounted wealth into a period by period summmation. The formulation with acceptance sets provides a setting for decomposing the multi-period constrained growth problem into a sequence of one period problems.

Proposition 1

Conditional on the optimal acceptance set, the optimal strategy at time t is path indepen-

dent, depending on the wealth at time t but not the path to that wealth. The problem is a sequence of static one period problems conditioned on the wealth from the previous period.

Proof:

Let A^* be the optimal acceptance set and consider the associated problem

$$max_X \left\{ E \sum_{t=1}^{T} \left\{ ln(R(X(t)) - \gamma I_{\overline{A_t^*}}[ln(w^*(t)) - ln(W(t-1)) - ln(R(X(t)))] \right\} \right\}$$

Subject to

$$ln(R(\omega, X(t)) \geq ln(w^*(t)) - ln(W(\omega, t-1)), t = 1, \ldots, T, \omega \in A^*$$

$$X^\top(t)e = 1, t = 1, ..., T.$$

The Lagrangian for this problem $L(X, \lambda^*, A^*) =$

$$E \sum_{t=1}^{T} \left[\left\{ ln(R(X(t)) - \gamma I_{\overline{A_t^*}}[ln(w^*(t)) - ln(W(t-1)) - ln(R(X(t)))] \right\} \right] +$$

$$E \sum_{t=1}^{T} I_{A_t^*} \lambda_t^*(\omega) \left(ln(w^*(t)) - ln(W(t-1)) - ln(R(X(t))) \right). \tag{14}$$

The multiplier $\lambda_t^*(\omega) \geq 0$ is in the space of the Lesbegue integrable functions on Ω and is such that $max_X \{L(X, \lambda^*, A^*)\}$is equivalent to the above problem. With $L_t(X, \lambda^*, A^*) =$

$$E \left[ln(R(X(t)) - \gamma I_{\overline{A^*}}[ln(w^*(t) - ln(W(t-1) - ln(R(X(t))] \right] +$$

$$E \left[I_{A^*} \lambda_t(\omega) \left(ln(w^*(t)) - ln(W(t-1) - ln(R(\omega, X(t))) \right) \right], \tag{15}$$

then $L(X, \lambda^*, A^*) = \sum_{t=1}^{T} \{L_t(X, \lambda^*, A^*)\}$. It can be seen that the Lagrangian is a sequence of T expressions, each conditioned on the wealth outcome from the previous time period. That is, the decision $X(1)$ given the initial wealth $w(t_0)$ leads to wealth $W(1) = w(1)$, which is the wealth at the start of period $t = 2$. In the t^{th} period the Lagrangian maximization is equivalent to the one period problem

$$Max_{X(t)} \left\{E\left[ln(R(X(t))) - \gamma I_{\overline{A}^*}[ln(w^*(t)) - ln(w(t-1)) - ln(R(X(t)))]\right]\right\} \tag{16}$$

Subject to

$$ln(R(X(t)) \geq (ln(w^*(t))) - ln(w(t-1)), \omega \in A^* \tag{17}$$

$$X(t)^\top e = 1.$$

That is, the dynamic multiperiod problem is a sequence of static one period problems conditional on the wealth from the previous period. □

The sequence of one period problems is defined for the optimal acceptance set A^*. There is a sequence of one period problems with a probability constraint which is equivalent to the optimal acceptance set sequence. For acceptance sets $A, P(A) \geq 1 - \alpha$,

$$max_A \left[max_X \left\{ E \sum_{t=1}^{T} \left\{ ln(R(t)^\top X(t)) - \gamma I_{\overline{A}_t}[ln(w^*(t)) - ln(W(t-1)) - ln(R(X(t)))] \right\} \right\} \right] \tag{18}$$

Subject to

$$ln(R(X(t)) \geq (ln(w^*(t))) - ln(W(t-1)), \omega \in A$$

$$X^{\top}(t)e = 1, t = 1, ..., T.$$

is equivalent to .

$$max_X \left\{ E \sum_{t=1}^{T} \left\{ ln(R(t)^{\top} X(t)) - \gamma [ln(w^*(t)) - ln(W(t-1)) - ln(R(X(t))]^+ \right\} \right\} \tag{19}$$

subject to

$$Pr\left[ln(W(t-1)) + ln(R(X(t)) \geq ln(w^*(t))), t = 1, \ldots, T\right] \geq 1 - \alpha \tag{20}$$

$$X^{\top}(t)e = 1, t = 1, ..., T.$$

.

So the probabilistic constraint contains the optimization over acceptance sets.

For the wealth process $ln(W(t)) = ln(W(t-1)) + ln(R(X(t))$, the constraint

$$Pr\left[ln(W(t-1)) + ln(R(X(t)) \geq ln(w^*(t))), t = 1, \ldots, T\right] \geq 1 - \alpha$$

is the same as

$$1 - Pr\left[\vee_{t=1}^{T} \left(ln(W(t-1)) + ln(R(X(t)) < ln(w^*(t))\right)\right] \geq 1 - \alpha.$$

.

With $Pr\left[ln(R(X(t)) < ln(w^*(t) - ln(w(t-1))\right] \leq \alpha_t$, where $\sum_{t=1}^{T} \alpha_t \leq \alpha$, the one period problem with a probabilistic constraint is

$$Max\left\{E\left[ln(R(X(t)) - \gamma[ln(w^*(t)) - ln(w(t-1)) - ln(R(X(t))]^+\right]\right\} \tag{21}$$

subject to

$$Pr\left[ln(R(X(t)) > ln(w^*(t) - ln(w(t-1)\right] \geq 1 - \alpha_t \tag{22}$$

$$X^\top(t)e = 1, t = 1, ..., T.$$

The requirement $\sum_{t=1}^{T} \alpha_t \leq \alpha$ is still part of the problem specification. The choice of α is replaced by a sequence $\{\alpha_t, t = 1, ..., T\}$. The α_t may be determined sequentially as the actual wealth trajectory $\{w(t), t = 1, ..., T\}$ unfolds relative to the benchmark path $\{w^*(t), t = 1, ..., T\}$. If a priori the periods are the same, the one period constraint probabilities would be $\alpha_t = \frac{1}{t}\alpha$. This is analogous to the Bonferroni method for determining an overall (family) rate α and period specific rates α_t.

3.3 Functional Form of One Period Problem

The multiperiod capital growth problem is structured as a sequence of one period problems, but the probabilistic constraints could pose a problem for solution. However, the setup for rates of return as normal within regimes and a mixture of normals overall makes the problem more tractable.

Assume that the distribution over regimes is $(p_1, ..., p_m)$ and let the unconditional return be $R(X(t))$. The conditional rate of return given regime k in period t is multivariate normal with

$$ln(R_k(X(t))) = \left[X'(t)(\phi_k - re) + r - \frac{1}{2}X'(t)\Delta_k^2 X(t)\right] + X'(t)\Delta_k Z_k.$$

If $ln(R_k(X(t))) < ln(w^*(t)) - ln(w(t-1))$, then $[ln(w^*(t) - ln(w(t-1) - ln(R_k(X(t))]^+$

has the same probability law as as $ln(R_k(X(t))$, which is Gaussian.

Let $f_k(v), k = 1, ..., m$ be the normal density of $ln(R_k(X(t))$, the log-return given the regime is k. The unconditional distribution for log-returns is a mixture of normals $f(v) = p_1 f_1(v) + \ldots + p_m f_m(v)$. The chance constraint in the one period problem given $w(t-1)$, in terms of log-return, is

$$Pr\left[ln(R(X(t)) > ln(w^*(t)) - ln(w(t-1))\right] \geq 1 - \alpha_t,$$

or

$$\int_{-\infty}^{ln(w^*(t))-ln(w(t-1))} [p_1 f_1(v) + \ldots + p_m f_m(v)]dv \leq \alpha_t. \tag{23}$$

Of course $\int_{-\infty}^{ln(w^*(t)-ln(w(t-1))} f_k(v)dv = \int_{-\infty}^{z_k^*(X(t))} f^\star(z)dz$, where $f^\star$ is the standard normal and $z_k^*(X(t)) = \frac{[ln(w^*(t))-ln(w(t-1))]-\mu_k(t))}{\sigma_k(t)}$, where $\mu_k(t) = X'(t)(\phi_k - re) + r - \frac{1}{2}\sigma_k^2(t)$ and $\sigma_k^2(t) = X'(t)\Delta_k^2 X(t)$.

Let $G(X(t)) = \sum_{k=1}^m p_k \int_{-\infty}^{z_k^*(X(t))} f^\star(z)dz - \alpha_t$. So the deterministic constraint is

$$G(X(t)) \leq 0. \tag{24}$$

The objective can be similarly reformulated. The expected rate of return is

$$E\left[ln(R(X(t)) - \gamma[ln(w^*(t)) - ln(w(t-1)) - ln(R(X(t)))]^+\right] =$$

$$\sum_{k=1}^m p_k \cdot E(ln(R_k(X(t))-$$

$$\gamma \sum_{k=1}^{m} p_k \int_{-\infty}^{z_k^*(X(t))} (ln(w^*(t)) - ln(w(t-1)) - [E(ln(R_k(X(t)) + z \cdot \sigma(ln(R_k(X(t)))]) f^\star(z) dz. \tag{25}$$

Let $F(X(t)) =$

$$\sum_{k=1}^{m} p_k \cdot [\mu_k(t)] -$$

$$\gamma \sum_{k=1}^{m} p_k \int_{-\infty}^{z_k^*(t)} (ln(w^*(t)) - ln(w(t-1)) - [\mu_k(t) + z \cdot \sigma_k(t)]) f^\star(z) dz. \tag{26}$$

.

Then the one period problem is

$$P(g(t), \gamma, \alpha_t) : max \left\{ F(X(t)) | G(X(t)) \leq 0 \right\}.$$

This problem depends on the gap between starting wealth and the path target, $g(t) = ln(w^*(t)) - ln(w(t-1)$, as well as the penalty γ and the shortfall probability α_t. The multiperiod problem is a sequence such one period problems, where the gap in the next period is controlled by the settings (γ, α_t).

4 The Kelly Strategy and the Penalty on Shortfalls

The optimal capital growth strategy, called the Kelly Strategy, maximizes the unconditional growth rate of capital $\sum_{k=1}^{m} p_k \cdot E(ln(R_k(X(t)))$. This strategy has many attractive properties and has been dubbed *Fortunes Formula* (MacLean, Thorp and Ziemba (2010), MacLean, Thorp, Zhao and Ziemba (2011)). One downside of the Kelly strategy is the chance of large losses. A prime motivation for the path constraint and shortfall penalty is to control large losses. In this section, the Kelly strategy and modifications with the path approach are considered. The market structure where the price dynamics follow the Markov regime switching model is assumed to hold. The Markovian assumption is important for the decomposition into a sequence of one period problems . So knowing the previous period regime, the probability distribution over regimes in the current period is given by the transition matrix.

4.1 Kelly Strategy with Regimes

Consider the one period problem $max \sum_{k=1}^{m} p_k \cdot E(ln(R_k(X(t))) = max \sum_{k=1}^{m} p_k \cdot \left[\mu_k(t) - \frac{1}{2}\sigma_k^2(t)\right]$. For $i = 1, ..., n$, the first order conditions are $\frac{\partial}{\partial x_i} \sum_{k=1}^{m} p_k \cdot \left[\mu_k(t) - \frac{1}{2}\sigma_k^2(t)\right] = 0$. This gives

$$X^* = \left(\sum_{k=1}^{K} p_k \Delta_k^2\right)^{-1} \left(\sum_{k=1}^{K} p_k(\phi_k - re)\right),$$

where $X^* = \begin{pmatrix} x_1^* \\ \vdots \\ x_n^* \end{pmatrix}$ are the fractions invested in the n risky assets.

So the Kelly strategy for the one period problem has the familiar form (Merton 1972), with the rate of return and volatility averaged over regimes. Since the distribution over regimes in period t will depend on the regime in period $t-1$ and the transition probabilities, the Kelly strategy will be current regime dependent. It is noteworthy that the Kelly strategy does not depend on the wealth at the beginning of the period: $w(t-1)$. However, the *performance* of the Kelly strategy relative to the path does depend on the starting wealth. In the risk context, performance is defined by the shortfall rate and the average shortfall size.

Given starting wealth $w(t-1)$ and the position relative to the path $g(t) = ln(w^*(t)) - ln(w(t-1)$, the *chance of a shortfall* in period t is $P[ln(R(X^*)) < g(t)] = \alpha_t^*$. Using the mixture of normals distribution,

$$\alpha_t^* = \int_{-\infty}^{g(t)} [\sum_{k=1}^{K} p_k f_k(v)] dv, \tag{27}$$

where $f_k(v)$ is the normal density with mean $Eln(R_k(X^*)) = X^{*'}(\phi_k - re) + r - \frac{1}{2} X^{*'} \Delta_k^2 X^*$

and variance $\sigma^2(ln(R_k(X^*)) = X^{*'}\Delta_k^2 X^*$.

The *average size of a shortfall* with the Kelly strategy, is

$$\eta_t^* = \frac{1}{\alpha^*}\left\{\int_{-\infty}^{g(t)} p_k[\sum_{k=1}^{K}(g(t)-v)f_k(v)]dv\right\}. \tag{28}$$

where $V_k = ln(R_k(X^*)$. The bi-criteria (α_t^*, η_t^*) can be combined into the ***risk score***

$$\varphi_t^* = \alpha_t^* \times \eta^*,$$

describing the risk relative to the path $w^*(t)$ and starting wealth $w(t-1)$.

To understand the risk associated with the Kelly strategy, the single risky asset case is developed.

CASE : Single Risky Asset

To simplify the analysis of shortfall rate and shortfall size with the Kelly strategy, consider the case of a single risky asset (the market index) and two regimes representing *UP* and *DOWN* markets. It is assumed the market is in a known regime and the transitions to $UP/DOWN$ in the coming period are (p_1, p_2). So the model parameters are $\Theta = (\phi_1, \delta_1^2, p_1, \phi_2, \delta_2^2, p_2)$. If $\overline{\phi} = p_1\phi_1 + p_2\phi_2$, and $\overline{\delta}^2 = p_1\delta_1^2 + p_2\delta_2^2$, the Kelly strategy invests $x^* = \frac{\overline{\phi}-r}{\overline{\delta}^2}$ in the risky asset. The *shortfall rate* is

$$\alpha_t^* = p_1\Phi(z_1^*) + p_2\Phi(z_2^*), \tag{29}$$

where Φ is the standard normal cumulative distribution and

$$z_1^* = \frac{g(t) - \left[x^*(\phi_1 - r) + r - \frac{1}{2}x^{*2}\delta_1^2\right]}{x^*\delta_1}$$

and

$$z_2^* = \frac{g(t) - \left[x^*(\phi_2 - r) + r - \frac{1}{2}x^{*2}\delta_2^2\right]}{x^*\delta_2}.$$

.

The *average shortfall size* is

$$\eta_t^* = \frac{1}{\alpha^*}\left\{p_1\int_{-\infty}^{g(t)}(g(t)-v)f_1(v)dv + p_2\int_{-\infty}^{g(t)}(g(t)-v)f_2(v)dv\right\}.$$

Let $\mu_k^* = x^*(\phi_k - r) + r - \frac{1}{2}x^{*2}\delta_k^2$ and $\sigma_k^* = x^*\delta_k$ for $k = 1, 2$.

Then $\int_{-\infty}^{g(t)}(g(t)-v)f_1(v)dv = g(t)\Phi(z_1^*) - \int_{-\infty}^{z_1^*}(\mu_1^* + z\sigma_1^*)f^\star(z)dz = [g(t)-\mu_1^*]\,\Phi(z_1^*) + \sigma_1^* f^\star(z_1^*)$. In the same way

$\int_{-\infty}^{g(t)}(g(t)-v)f_2(v)dv = [g(t)-\mu_2^*]\,\Phi(z_2^*) + \sigma_2^* f^\star(z_2^*)$. Combining terms gives

$$\eta_t^* = g(t) - \frac{1}{\alpha_t^*}\left\{p_1\left[\mu_1^*\Phi(z_1^*) - \sigma_1^*\Phi'(z_1^*)\right] + p_2\left[\mu_2^*\Phi(z_2^*) - \sigma_2^*\Phi'(z_2^*)\right]\right\} \tag{30}$$

The expressions for α_t^* and η_t^* are defined by the standard normal distribution Φ, the standard normal density $f^\star = \Phi'$, the mean μ_k and standard deviation σ_k of the return on the Kelly investment strategy in each regime and the transition probabilities (p_1, p_2). The Kelly strategy and investment returns depend on the base parameters $(\phi_1, \delta_1, \phi_2, \delta_2)$. With the interest in path shortfall, the DOWN parameters are important. Let $r = 0, \phi_1 = \phi, \delta_1 = \delta, p_1 = 1-p$, and $\phi_2 = (1-c)\phi, \delta_2 = (1+d)\,\delta$, for scaling constants (c, d). The constants define the DOWN returns relative to the UP returns. The down return rate and volatility can be analyzed separately with (i) $(d = 0) \Rightarrow (\alpha_t^*(c), \eta_t^*(c))$; (ii) $(c = 2) \Rightarrow (\alpha_t^*(d), \eta_t^*(d))$. The intention is to characterize the risk of the Kelly strategy as the downside gets more severe. The results below consider the downside risk from a decreasing rate of return in the DOWN regime, i.e as c increases.

For formulas the ratio $\psi = \frac{\phi}{\delta}$ is used. To simplify presentation, derivatives of key formula components are provided in Table 1.

Parameter	Formula	$\frac{\partial}{\partial c}$	$\frac{\partial^2}{\partial c^2}$
K_1	$1-cp$	$-p$	0
μ_1^*	$0.5K_1\psi^2\left[2-K_1\right]$	$-\psi^2 p\left[1-K_1\right]$	$-\psi^2p^2$
σ_1^*	$K_1\psi$	$-\psi p$	0
z_1^*	$\frac{g}{K_1\psi}-0.5\psi\left[2-K_1\right]$	$\frac{gp}{K_1^2\psi}-0.5\psi p$	$\frac{2gp^2}{K_1^3\psi}$
μ_2^*	$0.5K_1\psi^2\left[1-c(2-p)\right]$	$-0.5\psi^2\left[(K_1-c)(2-p)+1\right]$	$0.5\psi^2(2-p)(p+1)$
σ_2^*	$K_1\psi$	$-\psi p$	0
z_2^*	$\frac{g}{K_1\psi}-0.5\psi\left(1-c(2-p)\right)$	$\frac{gp}{K_1^2\psi}+0.5\psi(2-p)$	$\frac{2gp^2}{K_1^3\psi}$

Table 1: Kelly Portfolio Parameters: Rate of Return (c)

Proposition 2

Consider the single risky asset case with $\phi_1=\phi, \phi_2=(1-c)\phi, \delta_1=\delta_2=\delta$. If (α^*,η^*) are the (shortfall rate, expected shortfall size) for the Kelly strategy with initial gap g, then

(i) The rate of shortfall α^* accelerates as the downside return parameter c increases.

(ii) The expected shortfall size η^* accelerates as the downside return parameter c increases.

(iii) The acceleration in shortfall rate is greater that the acceleration in size.

Proof:

The Kelly strategy is $x^*=K_1\frac{\phi}{\delta^2}$. .With $\alpha^*=(1-p)\,\Phi(z_1^*)+p\Phi(z_2^*)$, then $\frac{\partial\alpha^*}{\partial c}=(1-p)\Phi'(z_1^*)\frac{\partial z_1^*}{\partial c}+p\Phi'(z_2^*)\frac{\partial z_2^*}{\partial c}>0$ and

$\frac{\partial^2\alpha^*}{\partial c^2}=(1-p)\left[\Phi''(z_1^*)\left(\frac{\partial z_1^*}{\partial c}\right)^2+\Phi'(z_1^*)\frac{\partial^2 z_1^*}{\partial c^2}\right]+p\left[\Phi''(z_2^*)\left(\frac{\partial z_2^*}{\partial c}\right)^2+\Phi'(z_2^*)\frac{\partial^2 z_2^*}{\partial c^2}\right]$. Since $\Phi''(z)=-z\Phi'(z)$, it follows that

$\frac{\partial^2\alpha^*}{\partial c^2}=(1-p)\,\Phi'(z_1^*)\left[\frac{\partial^2 z_1^*}{\partial c^2}-z_1^*\left(\frac{\partial z_1^*}{\partial c}\right)^2\right]+p\Phi'(z_2^*)\left[\frac{\partial^2 z_2^*}{\partial c^2}-z_2^*\left(\frac{\partial z_2^*}{\partial c}\right)^2\right]$. Using Table 1 gives $\frac{\partial^2\alpha^*}{\partial c^2}>0$ and the downside parameter is an accelerator for the shortfall rate with the Kelly strategy.

The expected shortfall size is $\eta^*=g-\frac{H(c)}{\alpha^*}$, where

$H(c)=(1-p)\left[\mu_1^*\Phi(z_1^*)-\sigma_1^*\Phi'(z_1^*)\right]+p\left[\mu_2^*\Phi(z_2^*)-\sigma_2^*\Phi'(z_2^*)\right]$.

So$\frac{\partial\eta^*}{\partial c}=\frac{\partial\alpha^*}{\partial c}\times\frac{\partial\eta^*}{\partial\alpha^*}>0$, since $\frac{\partial\eta^*}{\partial\alpha^*}=\frac{H(c)}{\alpha^{*2}}>0$. Also $\frac{\partial^2\eta^*}{\partial c^2}=\frac{\partial\alpha^*}{\partial c}\times\frac{\partial^2\eta^*}{\partial\alpha^{*2}}+\frac{\partial\eta^*}{\partial\alpha^*}\times\frac{\partial^2\alpha^*}{\partial c^2}<\frac{\partial^2\alpha^*}{\partial c^2}$.

As the downside regime degenerates, the performance of the Kelly strategy deteriorates both in terms of the shortfall rate and the expected shortfall size. Obviously the risk score φ accelerates as the rate of return deteriorates.

The chance of falling below the path and the expected size of the shortfall also depend on the starting position as given by the gap g.

Proposition 3

Consider the single risky asset case with $\phi_1 = \phi, \phi_2 = (1-c)\phi, \delta_1 = \delta_2 = \delta$. *If* (α^*, η^*) *are the (shortfall rate, expected shortfall size) for the Kelly strategy with initial gap* g, *then*

(i) The rate of shortfall α^* *decelerates as the gap* g *increases.*

(ii) The expected shortfall size η^* *decelerates as the gap* g *increases.*

(iii) The deceleration in shortfall rate is greater that the acceleration in size.

Proof:

Similar to Proposition 2.

The qualitative results consider the marginal effect of the expected rate of return in the down regime. The effect of simultaneous changes in the expectation and variance are considered numerically. In the computations the values $p_1 = 0.8, p_2 = 0.2, r = 0.00004, \phi_1 = .0003, \delta_1^2 = .00015, \delta_2^2 = .00015$ will be used. Consider, then, in Table 2 the *shortfall rate* and in Table 3 *average shortfall size* for a range of risky investment scenarios. The scenarios are defined by the relative rates in UP and DOWN regimes $c = 1 - \frac{\phi_2}{\phi_1}, \phi_1 > 0, \phi_2 < 0$, and the gap $g = w^* - w$. In the table the gap is shown as negative, that is the wealth at the beginning of the period is above the target. A shortfall occurs if the one period return is less than the gap.

The shortfall rate drops dramatically if the starting position is favorable relative to the path target. As the downside decreases the Kelly fraction also drops since the investment is less favorable.

The shortfall size as the regime parameter c and the gap above the path $-g$ change is provided in Table 3 for the same settings as Table 2. These numbers show the average size of the shortfall in terms of daily rate of returns, so the shortfalls are substantial. For example

c	ϕ_1	ϕ_2	x^*	-g				
				0.002	**0.006**	**0.010**	**0.014**	**0.018**
2.0	.0003	-.0003	.93	.43	.30	.19	.11	.06
2.2	.0003	-.00036	.85	.42	.28	.17	.09	.04
2.4	.0003	-.00042	.77	.41	.26	.14	.07	.03
2.6	.0003	-.00048	.69	.40	.24	.12	.05	.02
2.8	.0003	-.00054	.61	.39	.21	.09	.03	.01
3.0	.0003	-.00060	.53	.37	.18	.06	.02	.00
3,2	.0003	-.00066	.45	.35	.14	.04	.01	.00
3.4	.0003	-.00072	.37	.32	.09	.01	.00	.00
3.6	.0003	-.00078	.29	.28	.05	.00	.00	.00
3.8	.0003	-.00084	.21	.21	.01	.00	.00	.00

Table 2: Shortfall Rate: Kelly Strategy

in the situation where beginning wealth is close to the target ($-g = 0.002$), we have for shortfalls $\frac{W}{w^*} = 0.993$ or 99.3% of the target wealth on average. This level of fallback is equivalent to 17% of starting wealth on an annualized basis. The pattern in average size is similar to that for the shortfall rate , with the relative returns in UP and DOWN markets having a slight negative effect.

c	-g				
	0.002	**0.006**	**0.010**	**0.014**	**0.018**
2.0	0.0084	0.0072	0.0063	0.0055	0.0049
2.2	0.0076	0.0065	0.0056	0.0048	0.0042
2.4	0.0068	0.0057	0.0049	0.0042	0.0036
2.6	0.0061	0.0050	0.0042	0.0035	0.0031
2.8	0.0053	0.0043	0.0035	0.0029	0.0025
3.0	0.0045	0.0035	0.0028	0.0023	0.0020
3,2	0.0038	0.0028	0.0022	0.0018	0.0015
3.4	0.0030	0.0021	0.0016	0.0013	0.0010
3.6	0.0022	0.0015	0.0011	0.0008	0.0007
3.8	0.0015	0.0009	0.0006	0.0005	0.0004

Table 3: Shortfall Size: Kelly Strategy

4.2 Penalizing Shortfalls in the One Period Capital Growth Problem

The Kelly strategy can have an unacceptable risk of shortfalls, and that is the motivation for imposing a constraint on the rate and penalizing for the shortfall size. Since the utility of wealth is logarithmic, the approach is termed modified Kelly. In the case of a single risky asset, the investment proportion is a fractional Kelly, where the fraction is selected to satisfy both the rate constraint and the size penalty. The strategy will depend on the path and the starting wealth, as opposed to the pure Kelly.

The constrained one period problem problem is

$$max\left\{F(X(t))|G(X(t))\leq 0\right\}.$$

With x the fraction invested in the risky asset, $F(x)=\left(1-cp)\phi x-\frac{1}{2}x^2\delta^2\right)-\gamma\left(g(t)\left(p_1\Phi(z_1)+p_2\Phi(z_2)\right)\right)$ and $G(x)=p_1\Phi(z_1)+p_2\Phi(z_2)-\alpha$,where $\mu_k=x\left(\phi_k-r\right)+r-\frac{1}{2}x^2\delta_k^2$ and $\sigma_k=x\delta_k$, and $z_k=\frac{g(t)-\left[x(\phi_k-r)+r-\frac{1}{2}x^2\delta_k^2\right]}{x\delta_k}$ for $k=1,2$. Consider the Lagrangian

$$L(x,\lambda)=F(x)-\lambda G(x).$$

So

$$L(x,\lambda)=\left((1-cp)\phi x-\frac{1}{2}x^2\delta^2\right)-\left(\gamma g(t)+\lambda\right)\left(p_1\Phi(z_1)+p_2\Phi(z_2)\right)+$$

$$\gamma\left\{p_1\left[\mu_1\Phi(z_1)-\sigma_1\Phi'(z_1)\right]+p_2\left[\mu_2\Phi(z_2)-\sigma_2\Phi'(z_2)\right]\right\}-\lambda\alpha.$$

Let $x=fx^*$, where $x^*=argmax\left\{\left((1-cp)\phi x-\frac{1}{2}x^2\delta^2\right)\right\}$is the Kelly strategy. Since

$$\frac{\partial}{\partial x}\left(p_1\Phi(z_1)+p_2\Phi(z_2)\right)>0$$

and

$$\frac{\partial}{\partial x}\left\{p_1\left[\mu_1\Phi(z_1)-\sigma_1\Phi'(z_1)\right]+p_2\left[\mu_2\Phi(z_2)-\sigma_2\Phi'(z_2)\right]\right\}<0,$$

$max_x L(x,\lambda)$ is achieved when $(p_1\Phi(z_1)+p_2\Phi(z_2))=\alpha$, provided $L(x,\lambda)$ is increasing. Consider the case with no penalty: $\gamma=0$. So $L_0(fx^*,\lambda)=\left((1-cp)\phi x-\frac{1}{2}x^2\delta^2\right)-\lambda[(p_1\Phi(z_1)+p_2\Phi(z_2))-\alpha]$. Writing

$$G_0(f)=\left(p_1\Phi(z_1)+p_2\Phi(z_2)\right),$$

and $f_0=G_0^{-1}(\alpha)$, then $L_0(fx^*,\lambda)$ is increasing up to f_0x^*, and the optimal fraction is $min\left(f_0,1\right))$.

When the penalty is introduced, an additional term

$$\gamma\left\{p_1\left[\mu_1\Phi(z_1)-\sigma_1\Phi'(z_1)\right]+p_2\left[\mu_2\Phi(z_2)-\sigma_2\Phi'(z_2)\right]-g(t)\left(p_1\Phi(z_1)+p_2\Phi(z_2)\right)\right\}$$

is included in $L_\gamma(fx^*,\lambda)$, which is decreasing in the fraction f and for large penalty γ will dominate the term

$$\left((1-cp)\phi x-\frac{1}{2}x^2\delta^2\right)-\lambda[(p_1\Phi(z_1)+p_2\Phi(z_2))-\alpha].$$

The optimal fraction f_γ will be reduced from f_0. The strategy $f_\gamma x^*$is called *fractional Kelly.* and is optimal for the single risky asset case.

Consider the one period problem with starting wealth gap g and reliability level $\alpha=0.05$, where $p_1=0.8, p_2=0.2, r=0.00015$ and the values for daily return on stocks are $\phi_1=0.000375, \phi_2=-0.0005$ and $\delta_1^2=\delta_2^2=0.000225$. For this example the Kelly strategy is to invest the fraction $x^*=0.22$ in stock. Recall that $g=w^*(t)-w(t-1)$, the wealth position relative to the path benchmark. Table 4 gives the investment fraction in stock for a range of values for the parameter g and penalty parameter γ.

	-g							
γ	**.016**	**.014**	**.012**	**.010**	**.008**	**.006**	**.004**	**.002**
0	1.0	1.0	0.99	0.91	0.78	0.57	0.50	0.49
2	1.0	0.98	0.92	0.81	0.67	0.52	0.36	0.19
10	1.0	0.95	0.86	0.74	0.61	0.47	0.32	0.17

Table 4: Kelly Fractions

If starting wealth is close to the path target, the one period investment strategy is conservative when the shortfall rate constraint is imposed. A higher starting position leads to investment closer to the full Kelly strategy. In the growth framework it is never optimal to invest more than the full Kelly because growth falls and risk increases. The effect of the penalty on the investment fraction is significant.

5 Multiperiod Problem

The capital growth problem with path shortfall conditions is a multiperiod problem which is decomposed into a sequence of one period problems. The analysis of the Kelly strategy in Section 4 is based on the one period problem. It is clear that introducing the path through the gap parameter g has a substantial impact on the single period investment strategy. The gap is the linking condition between periods in the multiperiod problem. To observe the pattern in the gap between current wealth and the path target and its effect on the sequence of investment decisions and accumulated wealth, a multiperiod problem is solved as a linked sequence of one period problems. There is a risk free asset and two risky assets, stocks and bonds, with the investment fractions in period t being (x_0, x_1, x_2). It is assumed there is initial wealth of $W(0) = \$100$ and three states in the financial market; bull, transition, bear. The states are driven by a Markov switching process. The risky returns are considered to be lognormal, with parameter settings for asset price dynamics as follows:

- Annual rate of return on stocks and bonds $\phi_j, j = 1, 2, 3$, respectively:

$$\begin{pmatrix} -0.1 \\ 0.035 \end{pmatrix}, \begin{pmatrix} 0.015 \\ 0.06 \end{pmatrix}, \begin{pmatrix} 0.25 \\ -0.02 \end{pmatrix}.$$

- Covariance of returnsΔ_j^2 by regime $j = 1, 2, 3$,respectively:

$$\begin{pmatrix} 0.09 & -0.0108 \\ -0.0108 & 0.0324 \end{pmatrix}, \begin{pmatrix} 0.0484 & 0.0099 \\ 0.0099 & 0.0225 \end{pmatrix}, \begin{pmatrix} 0.0025 & -0.0015 \\ -0.0015 & 0.01 \end{pmatrix}.$$

- Transition matrix for regimes:

$$P = \begin{pmatrix} 0.15 & 0.75 & 0.1 \\ 0.1 & 0.8 & 0.1 \\ 0.2 & 0.1 & 0.7 \end{pmatrix}.$$

- Initial probability for regimes:

$$p(0) = (0, 1, 0).$$

- Annual Risk free rate:

$$r = 0.02.$$

- The target path will be developed sequentially depending on the status with respect to the path in the previous period. If there is a shortfall, so that $w(t-1) < w^*(t-1)$, then $w^*(t) = 1.01w(t-1)$. If the target is exceeded , so that $w(t-1) > w^*(t-1)$, then $w^*(t) = 0.99w(t-1)$. The intention is to relax the path requirement if the results are postive, thereby taking more risk.

The investment decisions are made daily for a period of one year - $T = 256$. At each time point the one period problem with the VaR constraint and penalty on path violations is

$$Max \left\{ E \left[ln(R(X(t)) - \gamma[ln(w^*(t)) - ln(w(t-1)) - ln(R(X(t)))]^+ \right] \right\}$$

subject to

$$Pr\left[ln(R(X(t))) > ln(w^*(t)) - ln(w(t-1))\right] \geq 1 - \alpha_t$$

$$X^\top(t)e = 1, t = 1, ..., T.$$

With $z_k^*(X(t)) = \frac{[g(t)] - \mu_k(t))}{\sigma_k(t)}$ the problem can be written as

$$max_{X(t)} \left\{F(X(t), \gamma, g(t)) | G(X(t), \alpha, g(t)) \leq 0\right\},$$

where $F(X(t), \gamma, g(t)) =$

$$\sum_{k=1}^{m} p_k \cdot [\mu_k(t)] -$$

$$\gamma \sum_{k=1}^{m} p_k \int_{-\infty}^{z_k^*(X(t))} (g(t) - [\mu_k(t) + z \cdot \sigma_k(t)]) f^\star(z) dz,$$

and $G(X(t), \gamma, g(t)) = \sum_{k=1}^{m} p_k \int_{-\infty}^{z_k^*(X(t))} f^\star(z) dz - \alpha_t$.

This is a nonconvex problem and a Monte Carlo approach is used to get the solution. (Mockus, 1989.)

In the one period problem at the start of period t , the values $(p_1, \ldots, p_K)$ are transition probabilities from the regime at time $t-1$ to the K possible regimes at time t. The transition probabilities and the asset return parameters are know in the problem studied here. In practice the parameters and probabilities are estimated. The EM algorithm (Dempster, 1977; Hamilton, 1989) is used to estimate those values in the hidden Markov model from the history of returns on risky assets.

To test the decision model, 5000 trajectories of 256 trading days are generated from the returns distributions. Along a trajectory the single period decision algorithm is implemented sequentially. A variety of performance statistics are calculated:

1. The violation probability = the relative frequency with which the wealth at the end of a period drops below the path target for that period. That is an average over the multiple decision periods.

2. The average final wealth = the average over the 5000 trajectories of the wealth after 256 trading days.

3. The Sharpe ratio for final wealth = the average risk adjusted return divided by the standard deviation.

4. The average cumulative shortfall = the average over the 5000 trajectories of the total shortfall along a trajectory.

Performance statistics for a selection of risk control settings are presented in Table 5.

Measure	Alpha	Gamma		
		0	2	10
Violation Probability	0.05	0.1482	0.0183	0.0040
	0.25	0.3504	0.0186	0.0037
	0.50	0.4100	0.0154	0.0040
Av Final Wealth	0.05	113.0546	106.2743	105.1149
	0.25	118.8159	106.5073	105.3578
	0.50	121.4290	106.2784	105.0543
Sharpe Ratio	0.05	0.5229	0.5558	0.5862
	0.25	0.5156	0.5328	0.6590
	0.50	0.5408	0.5866	0.5940
Av Cum shortfall	0.05	-70.7486	-7.9375	-2.0899
	0.25	-177.6402	-8.3683	-1.7586
	0.50	-243.5916	-5.6936	-1.7966

Table 5: Performance Statistics

The $\gamma = 0$ numbers are for the investment model with the path VaR constraint, but no penalty on shortfalls. The weak constraint with $\alpha = 0.50$ has a lot of downside. Even the stricter condition with $\alpha = 0.05$ still has a high shortfall rate and high average cumulative shortfall.

When the penalty on shortfalls is introduced, the effect is quite dramatic. Both the rate and aveage size are decreased substantially. The largest penalty $\gamma = 10$ has negligible rate and average size, and the average final wealth is comparable to that of the moderate penalty of $\gamma = 2$.

To indicate the effect of the risk control settings (alpha, gamma) on decisions the monthly average weights in stocks, bonds and cash are given in Figure 1 for values of $\alpha$and γ. The pattern in weights for the largest values are not shown, but are similar to the $\gamma = 2$ figures. When the penalty is introduced into the objective, the fraction in stocks declines substantially, with a corresponding increase in the fraction in the risk free asset. It is noteworthy that total investment in risky assets changes with the control settings, and also the relative fraction of the risky investment in stocks and bonds changes. The solution in the multiple risky asset case is not fractional Kelly per se, but it is an optimal growth strategy subject to control conditions.

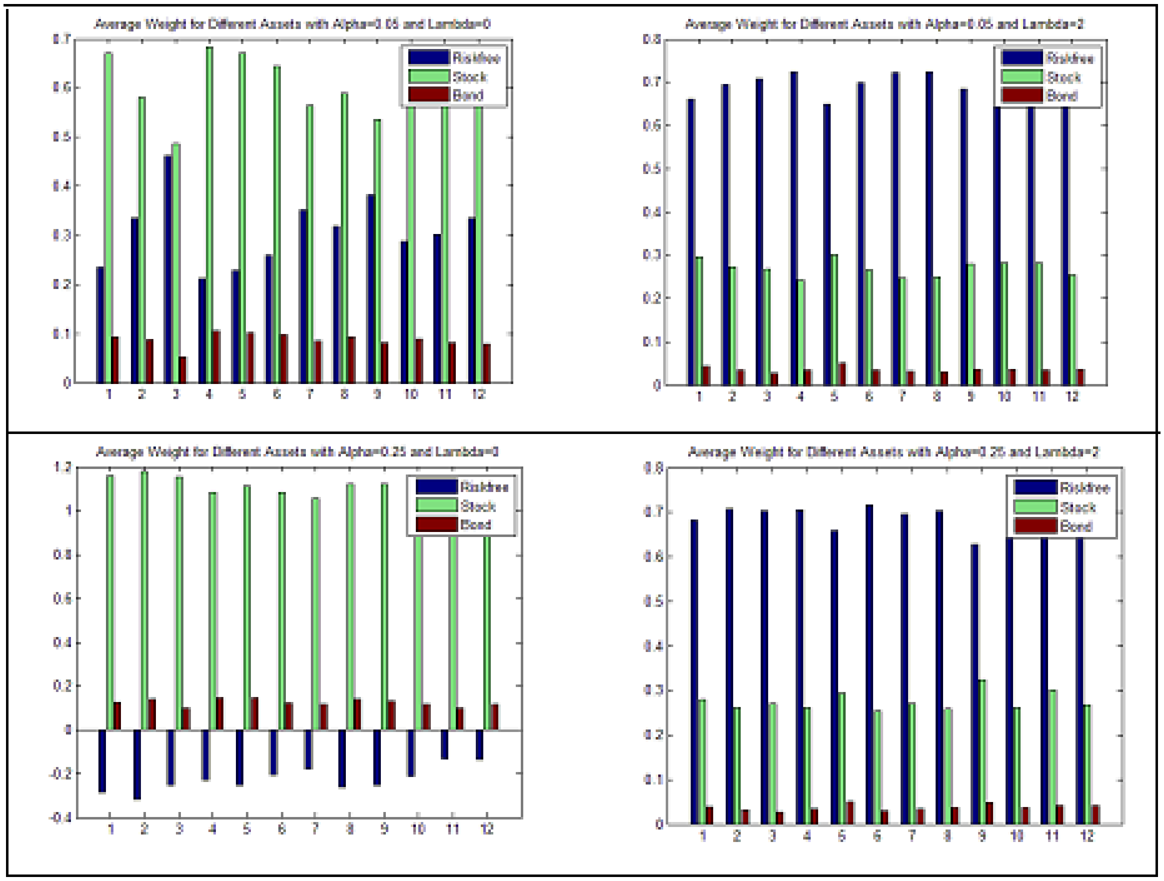

Figure 1: Investment Fractions

The following Figure 2 displays a sampling of 10 trajectories for values of $\alpha$and γ. The sample trajectories for the largest values are not shown, but are similar to the $\gamma = 2$ figures. Again the effect of the penalty is strong. The downside is controlled and the majority of trajectories experience positive growth at each time period.

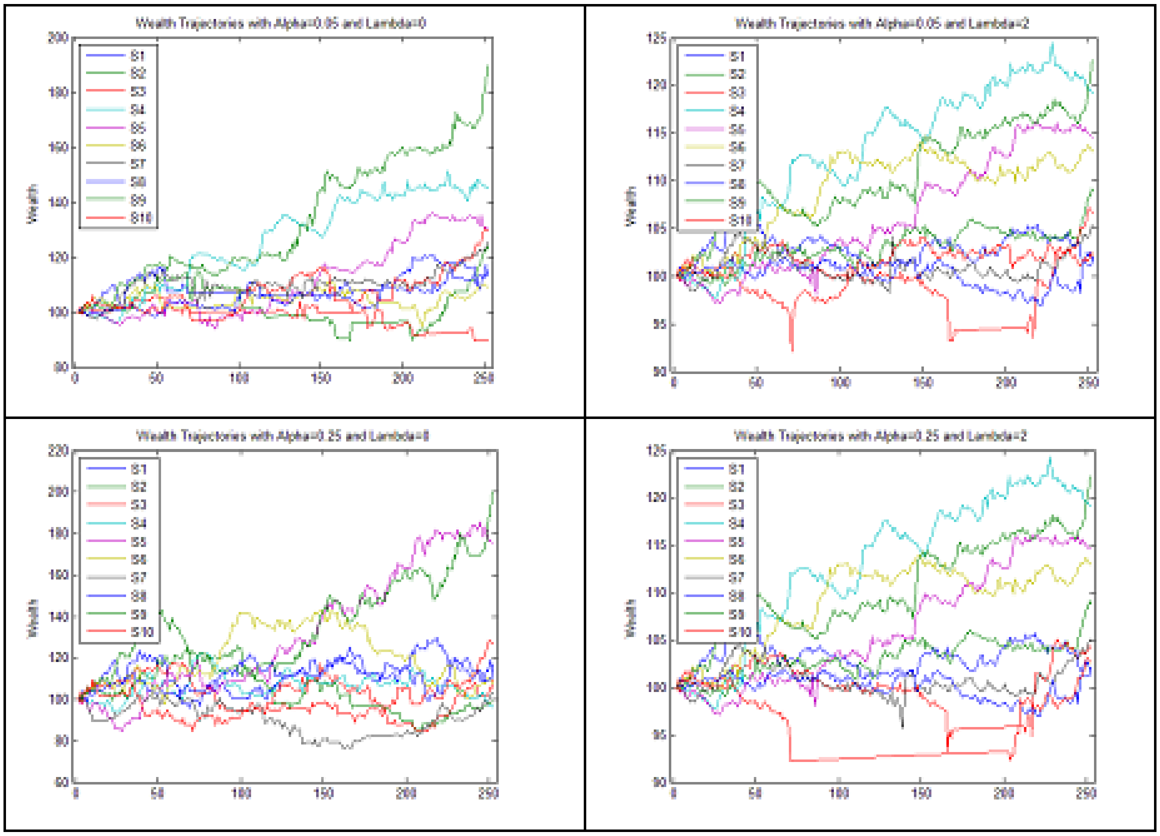

Figure 2: Wealth Trajectories

6 Conclusion

In the optimal capital growth problem where the rate of return to the horizon is maximized, the solution is often very aggressive and the chance of significant loss of capital in the short to medium term is too large. Putting a VaR constraint on the wealth trajectory controls the risk of losses, but the size of losses is still an issue. In this paper both the chance and size of losses is controlled. The loss shortfall is penalized in the objective with a wealth discounting approach. This retains the geometric character of the wealth process, or equivalently the arithmetic character of log-wealth. The model parameters are the VaR level, the VaR probability α, and the shortfall penalty γ. The impact of the parameters on strategies and accumulated capital is studied analytically with one risky asset (security) and a riskless asset. The methodology is also applied to the fundamental problem of investing in stocks and bonds over time. The convex penalty has the advantage of smoothing the trajectory of acculated capital while achieving capital growth. Excessive penalization of shortfalls leads to a path with little volatility, but it falls below low penalty paths along the full trajectory.

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