Report of the group
"p-adic methods in Arakelov geometry and Shimura varieties"

...

Junior Hausdorff Trimester Program

Algebraic Geometry

Period of stay: January - April 2014

1. The group

In 2013 a group of people got together and thought that it might be a very good idea to create an arena for various questions in the intersection of the theory of Shimura varieties, Arakelov geometry and p-adic geometry. These were Dennis Eriksson, Gerard Freixas i Montplet, Marc-Hubert Nicole and Siddarth Sankaran. After having been granted the scholarship, we were further enriched by Giuseppe Ancona who works on motives. All of us stayed for the entire duration of the programme, except Marc-Hubert Nicole who stayed for 3 months.

2. Organized activities at HIM

- Workshop in *Arithmetic intersection theory and Shimura varieties*.
- Mini-course on *p-adic geometry* by Jérôme Poineau.
- Mini-course on *Shimura varieties* by Marc-Hubert Nicole.
- Mini-course on *Motives* by Giuseppe Ancona.
- Organized the semi-weekly seminar PAS (12 speakers)

3. Research themes

One of the main objectives of our proposal was to gather together researchers with diverse backgrounds but common interests. While the main theme can be interpreted as arithmetic geometry, the more specific interests were in Arakelov geometry (intersection theory in the arithmetic sense, the intersections produce real numbers) and Shimura varieties (the geometric study of modular forms), with a particular view towards phenomena involving prime numbers p. The specific projects comprising our research proposal, are much motivated to some degree by Kudla's programme. Inspired by seminal work of Hirzebruch and Zagier, it is a deep conjectural framework which seeks to relate generating series of 'special' cycles on Shimura varieties with q-expansions of modular forms. Naturally, Arakelov geometry plays a key role, and most of the projects we describe either illuminates, extends, or draws analogy with, some aspect of the techniques involved in Kudla's programme. A more specific overview:

p-adic Kudla programme [N.] Inspired by Kudla's philosophy, Marc-Hubert has been investigating the arithmetic geometry of some explicit theta lifts especially in their p-adic avatars, with the hope of developing what he envisions as an emerging p-adic Kudla program. In a toy setup ("liftings from GL_2"), it replaces: Eisenstein series by explicit liftings of any classical modular form f such as the Saito-Kurokawa lifting (or the Shintani-Shimura-Waldspurger lifting; other theta liftings, etc.) in the context of p-adic families of modular forms; and the classical derivative by the p-adic derivative of the weight varying p-adically. One can propose to view as a special case of a p-adic Kudla program a collection of rather recent results combining p-adic techniques: p-adic families of modular forms, p-adic functoriality à la Langlands and algebraic cycles over local fields.

Riemann-Roch in Arakelov geometry and Jacquet-Langlands correspondance [E., F., S.] We have been exploring the relation between two kind of results in different branches of arithmetic geometry. The first one is the Riemann-Roch formula in Arakelov geometry. The second one is the Jacquet-Langlands correspondence in the theory of automorphic forms. In their common range of application, both results are related to the trace formula. For instance, the Riemann-Roch formula involves global invariants of automorphic forms (like eigenvalues under laplacian, Hecke eigenvalues), and the Jacquet-Langlands correspondence establishes relations between such invariants, for automorphic forms for inner forms of a given reductive group. Using a conjectural version of the arithmetic Riemann-Roch formula for open varieties, in the Hilbert modular case, we show that two (arithmetic) intersection numbers on varieties related by Jacquet-Langlands correspondance should coincide. The two numbers have only recently appeared in the literature, and our computations give an positive answer to this consequence of the conjectural Riemann-Roch formula. We also independently compute various other intersection numbers.

Realizing Riemann-Roch via theta functions [E., F.] The Riemann-Roch formula, in Grothendieck's formulation, gives the equality of certain characteristic classes associated to a fibration. In the particular case of fibration of curves, it has been known since some time, that some of these classes can be lifted to natural line bundles, and that the equality can be replaced by an isomorphism. Most of the methods on the market constructing the isomorphism are either entirely non-explicit, or slightly wrong. In this project we use theta functions and discriminants to realize the isomorphism in an explicit fashion. The current plan is to publish an extended version of this in book-form, but we are still awaiting a confirmation from the editors.

Asymptotic of fiber integrals [E. , F., Christophe Mourougane] For a fibration with compact complex manifold fibers of dimension n, $X \to D^*$, there are several studies of the asymptotics of fiber integrals as one approaches singular fibers. In this project we study the problem of integrating $\eta_t \wedge \overline{\eta_t}$, where η is a relative $(n, 0)$-form and let $t \to 0$ where 0 is a singular fiber. We relate these asymptotic formulas with well-known invariants from the mixed Hodge structure on the general fiber, and apply it to find analytical approaches to Kodaira-type canonical bundle formulas.

Pure motives attached to modular forms [A.] Scholl constructed motives associated with modular forms, lifting Deligne's construction of Galois representations attached to modular forms. This is done by studying the cohomology and the motive of modular curves (and the universal family of elliptic curves over them). A program, initiated by Wildeshaus, aim to generalize such constructions to all PEL Shimura varieties.

The Galois representations constructed by Deligne (as well as the motives constructed by Scholl) are *pure*. This is essential, for exemple, for Deligne's proof of Ramanujan's conjecture on the τ-function. On the other hand these modular forms live in the cohomology of varieties that although smooth are not compact (such as modular curves). Hence to understand purity one has to deal with *boundary cohomology*. In a preprint we settle some new cases.

Written works:

As a consequence of the 10h lecture series on Shimura varieties that N. gave at the H.I.M. in January 2014, he wrote a book chapter on the classical theory of unitary Shimura varieties to appear in the "Formes automorphes" book project directed by Michael Harris (IMJ, Columbia Univ.):

- Nicole, M.-H., Unitary Shimura Varieties, book chapter to appear in *Shimura Varieties and their associated Galois Representations* , Paris book project, vol. II, preprint, 25 pages, Oct. 2014.

- Giuseppe Ancona, *Degenerations of Hodge structures over Picard modular surfaces*, 15 pages, submitted.

- The other works mentioned are to appear in preprint form in a near future.

4. Working conditions

The set-up of the Institute made possible 10h long undisturbed working days. Indeed, the now famous 4 pm Institute cake would allow us not only to mingle and socialize with other participants, but also to get easy calories intake to last till a late supper, therefore adding 2-3h of work time to the day. On a related note, it was also very convenient to have simple access to bikes and discounts on several gyms in town.

Giuseppe Ancona, email: monsieur.beppe@gmail.com
Dennis Eriksson, email: dener@chalmers.se
Gerard Freixas i Montplet, email: gerard.freixas@imj-prg.fr
Marc-Hubert Nicole, email: marc-hubert.nicole@univ-amu.fr
Siddarth Sankaran, email: sankaran@math.toronto.edu