Sharp Isoperimetric Inequalities under $CDD(\rho, N, D)$

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Setup - Weighted Riemannian Manifold

- (M^n, g) Riemannian manifold: smooth, connected; d - induced geodesic distance.
- M is geodesically convex.
- $\mu = \Psi \cdot \mathrm{Vol}_g$, so that $\Psi \in C^2$ and $\Psi = \exp(-V) > 0$ on M.

Def (Bakry–Émery generalized Ricci tensor): given $N \in (-\infty, \infty]$

$$\mathrm{Ric}_{g,\mu,N} := \mathrm{Ric}_g - \mathrm{LogHess}_{N-n}\Psi$$

$$\mathrm{LogHess}_q \Psi = \nabla_g^2 \log \Psi + \frac{1}{q} \nabla_g \log \Psi \otimes \nabla_g \log \Psi = q \frac{\nabla_g^2 \Psi^{\frac{1}{q}}}{\Psi^{\frac{1}{q}}}.$$

- $N = n \Rightarrow \Psi \equiv c$, $\mathrm{Ric}_{g,\mu,n} = \mathrm{Ric}_g$.
- $N = +\infty \Rightarrow \mathrm{Ric}_{g,\mu,\infty} = \mathrm{Ric}_{g,\mu} = \mathrm{Ric}_g + \nabla_g^2 V$ (Lichnerowicz).

Def: (M^n, g, μ) satisfies Curvature-Dimension condition CD(ρ, N) if:

$$\exists \rho \in \mathbb{R} \qquad \mathrm{Ric}_{g,\mu,N} \geq \rho g \ \text{ on } M.$$

Remark: equivalent to original B–E def. for $N \in (-\infty, 0) \cup [n, \infty]$.

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Integrable 1-D localization preserves $CD(\rho, N)$

Let S denote C^2 hypersurface in (M^n, g) with unit normal field ν.

Set $F_S(x,t) := \exp_x(t\nu_x)$ defined for $(x,t) \in Dom(F_s) \subset S \times \mathbb{R}$.

Let $L_x \subset \mathbb{R}$ denote injectivity interval, i.e. maximal open interval containing origin s.t. $\forall t \in L_x \;\; d(F_S(x,t), S) = |t|$.

Let $J_S(x,t)$ be Jacobian of $F_S : (S, vol_S) \times (\mathbb{R}, dt) \to (M, vol_g)$.

Well-known:

- F_S is injective on $inj(S) := \{(x,t) \in Dom(F_S) \; ; \; t \in L_x\}$.
- $vol_g(Cut(S)) = 0$, where $Cut(S) = Im(F_S) \setminus inj(S)$.
- $J_S > 0$ on $inj(S)$, i.e. F_S is diffeomorphism there.

Hence by change-of-variables:

$$\forall \varphi \quad \int_{Im(F_S)} \varphi(y) dvol_g(y) = \int_S \int_{L_x} \varphi(F_S(x,t)) J_S(x,t) dt \, dvol_S(x)$$

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i.e. $J_{S,\mu} = J_S J_W$, where $J_W(x,t) = \frac{\Psi(F_S(x,t))}{\Psi(x)}$, s.t.:

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In other words we obtain an (integrable) 1-D localization:

$$\mu|_{Im(F_S)} = \int_S \mu_x dvol_{S,\mu}(x) \; , \; \mu_x = J_{S,\mu,x}(t) dt \; \text{ on } F_S(x,\cdot) : L_x \to M \; .$$

Theorem (Generalized Heintze–Karcher $N \in (-\infty, 1) \cup [n, \infty]$; HK $N = n$, Bayle $N \in (n, \infty)$, Morgan $N = \infty$, M. $N \in (-\infty, 1)$)

Let $(M^n, g, \mu) \in CD(\rho, N)$.
Then for all $x \in S$, $(L_x, |\cdot|, J_{S,\mu,x}(t) dt) \in CD(\rho, N)$, i.e.:

$$-LogHess_{N-1} J_{S,\mu,x}(t) \geq \varphi \quad \text{on } L_x .$$

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Proof of Generalized Heintze–Karcher

Fix $x \in S$, set $\nu := \nabla_t F_s(x,t)$ and recall $J = J_S J_W$, $J_W = \frac{\Psi(F_S(x,t))}{\Psi(x)}$.

$$-(\log J_S)'' - \frac{1}{n-1}((\log J_S)')^2 \geq Ric_g(\nu,\nu) \ ,$$

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Summing and applying Cauchy–Schwarz for $\{\alpha, \beta > 0\}$ ($N \in [n,\infty]$) or $\{\alpha\beta < 0 \text{ and } \alpha + \beta < 0\}$ ($N - 1 \in (-\infty, 0)$):

$$\frac{A^2}{\alpha} + \frac{B^2}{\beta} \geq \frac{(A+B)^2}{\alpha+\beta} \quad \forall A, B \in \mathbb{R},$$

we obtain the desired:

$$-\mathrm{LogHess}_{N-1} J = -(\log J)'' - \frac{1}{N-1}((\log J)')^2 \geq \mathrm{Ric}_{g,\mu,N}(\nu,\nu) \geq \rho \qquad \square$$

Remark: we proved for (α, β) as above:

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Isoperimetric Inequalities - Definitions

Metric-Measure Space: (Ω, d, μ), (Ω, d) Polish, μ Borel.
In our weighted manifold setting $(\Omega, d, \mu) = (M, g, \mu)$.

Isoperimetric Inqs compare between $\mu(A)$ and $\mu^+(A)$ (Minkowski's exterior boundary measure):

$\mu^+(A) := \liminf_{\varepsilon \to 0} \frac{\mu(A^d_\varepsilon) - \mu(A)}{\varepsilon}$,

$A^d_\varepsilon := \{x \in \Omega ; d(x, A) < \varepsilon\}$.

Isoperimetric profile: $\mathcal{I} = \mathcal{I}(\Omega, d, \mu) : [0, \mu(\Omega)] \to \mathbb{R}_+$ defined:

$$\mathcal{I}(v) := \inf \left\{ \mu^+(A) \; ; \; \mu(A) = v \right\} \; , \; \mu^+(A) \geq \mathcal{I}(\mu(A)) \; .$$

On $(\mathbb{R}, |\cdot|, \mu)$, also define flat profile $\mathcal{I}^\flat = \mathcal{I}^\flat(\mathbb{R}, |\cdot|, \mu)$:

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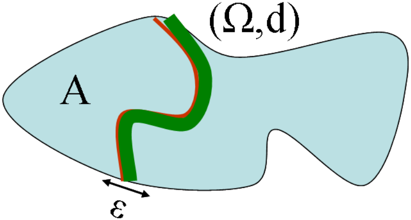

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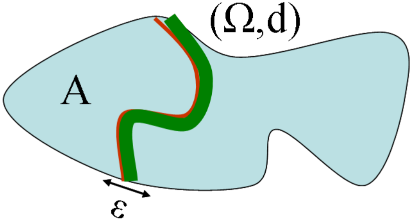

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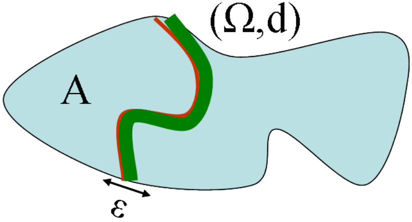

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Existence & regularity of **isop minimizers** (Geometric Measure Th.: Almgren, Bombieri, DeGiorgi, Federer, Fleming, Giusti, Gonzalez–Massari–Tamanini, Morgan, Simons):

- $\overline{\partial A \cap \mathring{M}} = \partial_s A \cup \partial_r A$; $\partial_s A$ - singular (Hausdorff dim $\leq n - 8$), $\partial_r A$ regular hypersurface (as smooth as Ψ), CMC: $H_{\partial_r A, \mu} \equiv H_\mu(A)$.
- Normal rays from $\partial_r A$ sweep out entire $\mathring{M} \setminus \partial_s A = Im(F_{\partial_r A})$ (when $\partial M = \emptyset$, due to Gromov '80).
- Discussion.

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Existence & regularity of **isop minimizers** (Geometric Measure Th.: Almgren, Bombieri, DeGiorgi, Federer, Fleming, Giusti, Gonzalez–Massari–Tamanini, Morgan, Simons):

- $\overline{\partial A \cap \mathring{M}} = \partial_s A \cup \partial_r A$; $\partial_s A$ - singular (Hausdorff dim $\leq n - 8$), $\partial_r A$ regular hypersurface (as smooth as Ψ), CMC: $H_{\partial_r A, \mu} \equiv H_\mu(A)$.
- Normal rays from $\partial_r A$ sweep out entire $\mathring{M} \setminus \partial_s A = Im(F_{\partial_r A})$ (when $\partial M = \emptyset$, due to Gromov '80).
- Discussion.

Balanced 1-D localization induced by Isop Minimizer

Conclusion: every isoperimetric minimizer A induces a 1-D localization perpendicular to $S = \partial_r A$:

$$\mu = \int_S \mu_x dvol_{S,\mu}(x) \; , \; \mu_x = J_{S,\mu,x}(t)dt \; \text{ on } \; \exp_x(t\nu_S) : L_x \to M \; ,$$

which is gen-mean-curvature balanced: $\forall x \in S \; J'_{S,\mu,x}(0) = H_\mu(A)$.
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Draw Picture!

By Gen-Heintze-Karcher, if $N \in (-\infty, 1) \cup [n, \infty]$:

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A 1-D maximum principle

Lemma: Assume that on an interval L_1 containing origin:

$$-\text{LogHess}_{\mathcal{N}} J_1 = -\mathcal{N}\frac{(J_1^{\frac{1}{\mathcal{N}}})''}{J_1^{\frac{1}{\mathcal{N}}}} \geq \rho \; , \; J_1(0) = 1 \; , \; J_1'(0) = H.$$

Let J_0 be solution to the equality case with same boundary conds:

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on the maximal interval L_0 containing origin where $J_0 \in (0, \infty)$.

Then: ($\mathcal{N} > 0$) $L_1 \subset L_0$ and $J_1 \leq J_0$ on L_1 (for $\mathcal{N} < 0$ reverse L_1/L_0).

Proof: denote $k_i = J_i^{\frac{1}{\mathcal{N}}}$, $\mathcal{N} k_1'' + \rho k_1 \leq 0$, $\mathcal{N} k_0'' + \rho k_0 = 0$.
Set $q := k_1' k_0 - k_0' k_1$. Then $q(0) = 0$ and $\mathcal{N} q'(t) \leq 0$.
Hence $(\log J_1)' \leq (\log J_0)'$ on $t > 0$, $(\log J_1)' \geq (\log J_0)'$ on $t < 0$. □

Notation: $J_{H,\rho,\mathcal{N}} := ((J_0^{\frac{1}{\mathcal{N}}})_+)^{\mathcal{N}}$ (i.e. set $0/\infty$ outside L_0).
Cor: $(L_1, |\cdot|, J_1(t)dt) \in CD(\rho, N), J_1(0) = 0, J_1'(0) = H \Rightarrow$
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Explicit Description of $J_{H,\rho,\mathcal{N}}$

$J_{H,\rho,\mathcal{N}}$ coincides with solution J to following Riccati ODE, on maximal interval containing the origin where such a solution exists:

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Given $H, \rho \in \mathbb{R}$, $\mathcal{N} \in (-\infty, \infty] \setminus \{0\}$, set $K := \rho/\mathcal{N}$ and define:

$$J_{H,\rho,\mathcal{N}}(t) := \begin{cases} \left(\left(c_K(t) + \frac{H}{\mathcal{N}} s_K(t)\right)_+\right)^{\mathcal{N}} & \mathcal{N} \notin \{0, \infty\} \\ \exp(Ht - \frac{\rho}{2}t^2) & \mathcal{N} = \infty \end{cases} .$$

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Combining everything for $(M^n, g, \mu) \in CD(\rho, N)$

Given isoperimetric minimizer $A \subset M$, $\mu(A) = v$, $\mu^+(A) = \mathcal{I}(v)$, recall our $H_\mu(A)$-balanced 1-D localization:

$$\mu = \int_{\partial_r A} \mu_x dvol_{\partial_r A, \mu}(x) \ , \ (L_x, |\cdot| \, , \mu_x = J_{\partial_r A, \mu, x}(t)dt) \in CD(\rho, N) \ .$$

By 1-D max principle, $J_{\partial_r A, \mu, x} \leq J_{H, \rho, N-1}$ with $H = J'_{\partial_r A, \mu, x}(0)$, so:

$$\mu(F_{\partial_r A}(\partial_r A \times [0, b])) \leq \int_{\partial_r A} \int_0^b J_{H_\mu(A), \rho, N-1}(t) dt \ dvol_{\partial_r A, \mu}(x).$$

Hence: $\exists a + b \leq \mathrm{diam}(M) =: D \in (0, \infty]$:

$$1 - v = \mu(M \setminus A) \leq \mu((\partial_r A)^+_b) \leq \mu^+(A) \int_0^b J_{H_\mu(A), \rho, N-1}(t) dt \ ,$$

$$v = \mu(A) \leq \mu((\partial_r A)^-_a) \leq \mu^+(A) \int_0^a J_{-H_\mu(A), \rho, N-1}(t) dt \ .$$

$$\Rightarrow \quad \mu^+(A) \geq \inf_{H \in \mathbb{R}, a+b=D} \max \left(\frac{v}{\int_{-a}^0 J_{H, \rho, N-1}(t)}, \frac{1-v}{\int_0^b J_{H, \rho, N-1}(t)} \right) =: \mathcal{GL}^\flat_{\rho, N-1, D}(v).$$

Isoperimetric Inequality for CDD(ρ, N, D)

Assume CDD(ρ, N, D) = CD(ρ, N) & diam(M) $\leq D \in (0, \infty]$, $\rho \in \mathbb{R}$.

Thm (Isoperimetric Inequality for CDD, M. '12,'14)

1. If $N \in (-\infty, 1) \cup [n, \infty]$ then for all $v \in (0, 1)$:

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2. If $N \in (-\infty, 0] \cup [n, \infty]$ or $D = \infty$ then $\mathcal{GL}^\flat_{\rho,N-1,D} = \mathcal{I}^\flat_{\rho,N-1,D}$,

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where the inner infimum over an empty set of H's is 0.

Notation: interval $L \subset \mathbb{R}$, $\mathcal{I}^\flat(f, L) = \mathcal{I}^\flat(\mathbb{R}, |\cdot|, \frac{1}{\int_L f(x)dx} f(x) 1_L(x)dx)$.

Thm (Sharpness, M. '12)

Part (2) automatically yields sharpness for $n = 1$, $\forall v \in [0, 1]$. In fact, sharpness verified for all $n \geq 2$, $N \in [n, \infty]$, $v \in [0, 1]$, ρ and D.

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Denote the "generalized sectional-curvature" $K := \rho/(N-1)$.

Case S1 - $N \in [n, \infty)$, $\rho > 0$, $D \geq \pi/\sqrt{K}$ ($N = n$ recovers Lévy–Gromov, $N > n$ Bayle):

$$\mathcal{I}(M, g, \mu) \geq \mathcal{I}^\flat \left(\sin(\sqrt{K}t)^{N-1}, [0, \pi/\sqrt{K}] \right) \quad N\text{-dim sphere with Ric} = \rho \, .$$

Case S2 - $N = \infty$, $\rho > 0$, $D = \infty$ (recovers Sudakov–Tsirelson, Borell, Bakry–Ledoux):

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Case S3 - $N \in (-\infty, 1)$, $\rho > 0$, $D = \infty$ (new '14):

$$\mathcal{I}(M, g, \mu) \geq \mathcal{I}^\flat \left(\cosh(\sqrt{K}t)^{N-1}, \mathbb{R} \right) \quad \text{positively curved } N\text{-dim hyperbolic space} \, .$$

In all other cases, no single model space, but one-parameter family of 1-D spaces:

Case F1A - $N \in [n, \infty)$, $\rho > 0$, $D < \pi/\sqrt{K}$ ('12, improves Bérard–Besson–Gallot):

$$\mathcal{I}(M, g, \mu)(v) \geq \min_{\xi \in [0, \frac{\pi}{\sqrt{K}} - D]} \mathcal{I}^\flat \left(\sin(\sqrt{K}t)^{N-1}, [\xi, \xi + D] \right)(v) \, .$$

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Isoperimetric Inequalities for CDD(ρ, N, D) - Simplified

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Case F1B - $N \in (-\infty, 0]$, $\rho < 0$, $D < \pi/\sqrt{K}$ (new '14):

$$\mathcal{I}(M, g, \mu) \geq \inf_{\xi \in (0, \frac{\pi}{\sqrt{K}} - D)} \mathcal{I}^\flat \left(\sin(\sqrt{K} t)^{N-1}, [\xi, \xi + D]\right) .$$

Case F3AB - $\{N \in [n, \infty), \rho < 0\}$ or $\{N \in (-\infty, 0], \rho > 0\}$, $D < \infty$ (new '12,'14):

$$\mathcal{I}(M, g, \mu) \geq \min \left\{ \begin{array}{c} \inf_{\xi > 0} \mathcal{I}^\flat(\sinh(\sqrt{-K} t)^{N-1}, [\xi, \xi + D]) , \\ \mathcal{I}^\flat(\exp(\sqrt{-K} t)^{N-1}, [0, D]) , \\ \inf_{\xi \in \mathbb{R}} \mathcal{I}^\flat(\cosh(\sqrt{-K} t)^{N-1}, [\xi, \xi + D]) \end{array} \right\} .$$

Case F4 - $N \in (-\infty, 0] \cup [n, \infty)$, $\rho = 0$, $D < \infty$ (new '12,'14)

$$\mathcal{I}(M^n, g, \mu) \geq \min \left\{ \begin{array}{c} \inf_{\xi > 0} \mathcal{I}^\flat(t^{N-1}, [\xi, \xi + D]), \\ \mathcal{I}^\flat(1, [0, D])) \end{array} \right\} .$$

Case F5 - $N = \infty$, $\rho = 0$, $D < \infty$ ('12)

$$\mathcal{I}(M^n, g, \mu) \geq \min_{H \geq 0} \mathcal{I}^\flat(\exp(Ht), [0, D]) .$$