How Good is 1/n Portfolio?

Workshop on Stochastic Optimization – Models and Algorithms at Hausdorff Research Institute for Mathematics

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1/n versus Optimal Portfolio

- Conventional wisdom: 1/*n* portfolio, or formally, the equally weighted portfolio performs well.
  - Babylonian Talmud (see Duchin and Levy, 2009)

- Some reasoning
  - Kallberg and Ziemba (1976):
    - Mean errors are 20 or more times as important as covariance errors.
    - Substantial errors in mean-variance optimization can greatly reduce ex post portfolio performance
  - Jobson and Korkie (1980): equal-weighted portfolios might outperform mean-variance optimal portfolios due to these estimation errors
  - Hensel and Ziemba(1995) and Ziemba (2013):
    - Small cap stocks tend to outperform large cap stocks over long periods
    - 1/n is one type of small cap strategy.
  - Rebalancing gains, etc.

Debates

- Jagannathan and Ma (2003)
  - The minimum variance portfolio with non-negativity constraints has a higher Sharpe ratio than the 1/*n* portfolio
- DeMiguel et al. (2009)
  - Compare the 1/*n* portfolio with various optimal portfolio strategies in an out-of-sample setting
  - 1/*n* can have superior performance than mean-variance optimal portfolios
  - Most of these optimal portfolios fail to perform well in real data sets
- Kritzman, Page, and Turkington (2010)
  - Rebuttal to DeMiguel et al. (2009)
  - Results of DeMiguel et al. (2009) are overturned when long sample periods are utilized in estimation
- Duchin and Levy (2009)
  - Out-of-sample performance also depends on the number of assets under consideration where a smaller number favors the 1/*n* portfolio
- Tu and Zhou (2011)
  - Combine the 1/n rule with various versions of the optimal portfolio and achieve superior performance
- … and many more

Why Does Debate Continue?

- Typical approach for empirical test
  - Pick data sets
  - Construct portfolios
    - 1/n
    - Optimal portfolios based on many different estimation techniques
  - Compare performance

- Why does debate continue?
  - Issue 1: The result varies by the choice of data set
  - Issue 2: The result varies by the choice of estimation techniques for optimal portfolio

- Objective of this study: provide a framework to overcome Issue 2

A Closer Look on Issue 2

- For an asset universe with n securities
  - Number of all possible portfolios: uncountably many
  - Number of optimal portfolios compared against 1/n: finite

- Thus, Issue 2 is really about "how representative these optimal portfolios are".

- Remedy
  - Step 1: enumerate "all possible portfolios" in an asset universe
  - Step 2: find out what % of all portfolios are outperformed by 1/n
  - Step 3: if 1/n does not outperform at least half of all portfolios, conclude that 1/n is in fact not so great

- Main question: how to enumerate all possible portfolios?
  - We propose "a uniformly distributed random portfolio".

Assumptions

- Asset universe
  - Consists of n risky securities $r \in \mathbb{R}^n$ along with a risk-free asset $r_f \in \mathbb{R}$.
  - Shorting of risky securities is allowed
  - One can borrow and lend the capital at the risk-free rate
  - True parameters that cannot be obtained at t=0

$$\mathbb{E}\left[r - r_f \mathbf{1}\right] = \mu \in \mathbb{R}^n$$
$$\mathbb{V}\mathrm{ar}[r] = \mathbb{V}\mathrm{ar}\left[r - r_f \mathbf{1}\right] = \Sigma \in \mathbb{R}^{n \times n}$$

  - Σ is strictly positive definite, thus is invertible.

- Decision variables
  - w_f: capital to be allocated to the risk-free asset
  - Thus, $1 - w_f \in \mathbb{R}$ to be allocated to the risky securities
  - $w \in \mathbb{R}^n$ such that $w^T \mathbf{1} = 1$: weights on the risky securities

- Performance measure: Sharpe ratio

$$SR^0(w|\mu, \Sigma) = \frac{\mathbb{E}[w^T r|\mu, \Sigma] - r_f}{\sqrt{\mathbb{V}\mathrm{ar}[w^T r|\mu, \Sigma]}} = \frac{\mu^T w}{\sqrt{w^T \Sigma w}}$$

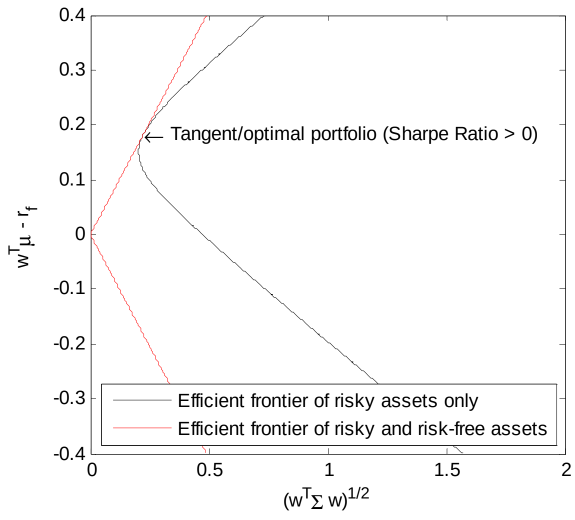
Best and Worst Portfolios

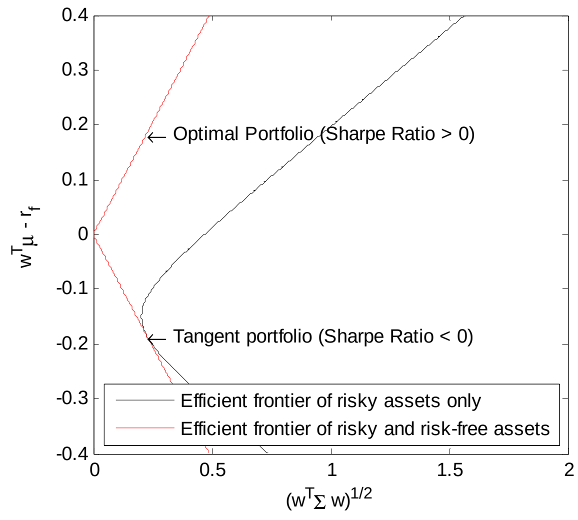
Ex-post *optimal* portfolio of risky securities

$$w^* = \mathrm{sgn}(\lambda) w^{tan} = \mathrm{sgn}(\lambda)\lambda\Sigma^{-1}\mu = |\lambda|\Sigma^{-1}\mu$$

Range of Sharpe ratio in this asset universe

$$[SR^0(-w^*|\mu,\Sigma), SR^0(w^*|\mu,\Sigma)] = \left[-\sqrt{\mu^T\Sigma^{-1}\mu}, \sqrt{\mu^T\Sigma^{-1}\mu}\right]$$





Portfolio Construction

Let $(\hat{\mu}, \hat{\Sigma})$ be the estimators of (μ, Σ) of a portfolio manager. Then, for $\hat{\lambda}$ satisfies $\mathbf{1}^T \hat{w}^{tan} = 1$, the *ex-ante tangent* portfolio

$$\hat{w}^{tan} = \underset{w \in \{w | \mathbf{1}^T w = 1\}}{\operatorname{argmax}} \frac{\hat{\mu}^T w}{\sqrt{w^T \hat{\Sigma} w}} = \hat{\lambda} \hat{\Sigma}^{-1} \hat{\mu}$$

Consequently, her optimal portfolio

$$\hat{w}^* = \operatorname{sgn}(\hat{\lambda}) \hat{w}^{tan} = \operatorname{sgn}(\hat{\lambda}) \hat{\lambda} \hat{\Sigma}^{-1} \hat{\mu} = |\hat{\lambda}| \hat{\Sigma}^{-1} \hat{\mu}$$

<u>*Definition*</u> Let $\hat{x} = \hat{\Sigma}^{-1} \hat{\mu}$ be *the aggregated estimator*.

Then, given the true parameters (μ, Σ), the Sharpe ratio of $\hat{w}^*$

$$SR^0(\hat{w}^* | \mu, \Sigma) = \frac{\hat{w}^{*T} \mu}{\sqrt{\hat{w}^{*T} \Sigma \hat{w}^*}} = \operatorname{sgn}(\hat{x}^T \mu) \sqrt{\frac{\hat{x}^T \mu \mu^T \hat{x}}{\hat{x}^T \Sigma \hat{x}}}$$

Estimating Parameters

Proposition Suppose the manager allocates $\left(1-w_f\right)$ of her capital to the risky securities according to the ex-ante optimal portfolio $\widehat{w}^*$, and allocates remaining capital w_f to the risk-free asset. If the allocation to the risky portfolio $\widehat{w}^*$ non-zero, that is $\left(1-w_f\right) \neq 0$, given the true parameters (μ, Σ), the Sharpe ratio of her portfolio of risk-free and risky securities is solely determined by the choice of $\widehat{w}^*$ and does *not* depend on the choice of w_f.

Proposition Given the true parameters (μ, Σ), the Sharpe ratio of the manager's ex-ante optimal portfolio of risky securities is determined by a single quantity $\hat{x}$, *not separately by* $\hat{\mu}$ and $\hat{\Sigma}$. In other words, it is enough for the manager to pick the value of $\hat{x}$ to construct the portfolio, and there is no need to specify the values of $\hat{\mu}$ and $\hat{\Sigma}$ separately at t = 0.

A Skilled Manager

Definition (Sharpe ratio function on aggregated estimator) Let $\hat{x}$ be the aggregated estimator of a manager. Then, the Sharpe ratio function of the portfolio of risky securities constructed with the estimator $\hat{x}$ given (μ, Σ) is:

$$SR(\hat{x}|\mu, \Sigma) \coloneqq SR^0(\widehat{w}^*|\mu, \Sigma) = \operatorname{sgn}(\hat{x}^T\mu)\sqrt{\frac{\hat{x}^T\mu\mu^T\hat{x}}{\hat{x}^T\Sigma\hat{x}}}.$$

Proposition For any $\gamma \in \mathbb{R}^{++}$, $SR(\hat{x}|\mu, \Sigma) = SR(\gamma\hat{x}|\mu, \Sigma)$. So, for fixed (μ, Σ), the Sharpe ratio of the manager's portfolio is scale-invariant with respect to $\hat{x}$.

Thus, we can restrict $\hat{x}$ to reside only on the surface of n-dimensional unit hypersphere.

A Skilled Manager

Definition A portfolio manager who constructs her portfolio by employing the following procedure is referred to as a skilled portfolio manager.

- At t = 0, she estimates $\hat{x} = \hat{\Sigma}^{-1}\hat{\mu}$ as a single quantity, and does not estimate $\hat{\mu}$ and $\hat{\Sigma}$ separately.

- She understands that her performance in Sharpe ratio is scale-invariant with respect to $\hat{x}$.

- Her portfolio $\hat{w}^* = \mathrm{sgn}(\hat{\lambda})\hat{w}^{tan} = \mathrm{sgn}(\hat{\lambda})\hat{\lambda}\hat{\Sigma}^{-1}\hat{\mu} = |\hat{\lambda}|\hat{\Sigma}^{-1}\hat{\mu}$

- Then, her performance

$$SR(\hat{x}|\mu, \Sigma) = \mathrm{sgn}(\hat{x}^T\mu)\sqrt{\frac{\hat{x}^T\mu\mu^T\hat{x}}{\hat{x}^T\Sigma\hat{x}}}$$

Q) What is the distribution of $SR(\hat{x}|\mu, \Sigma)$, if $\hat{x}$ is uniformly distributed on the surface of unit hypersphere?

A Uniformly Distributed Random Portfolio

Definition

A *uniformly distributed random portfolio* $\widehat{w}^{unif} \in \mathbb{R}^n$ is a portfolio constructed by a skilled manager who picks the value of the aggregated estimator $\hat{x}$ randomly so that any point on the surface of an n-dimensional unit hypersphere is equally likely to be assigned to $\hat{x}_\Sigma = L_\Sigma^T \hat{x}$ for L_Σ the Cholesky decomposition of Σ satisfying $\Sigma = L_\Sigma L_\Sigma^T$. In other words, $\hat{x}_\Sigma$ is a uniformly distributed random variable on the surface of the unit hypersphere*.

*Some remarks will be made at the end of presentation as to why we let $\hat{x}_\Sigma$ be uniformly distributed, rather than $\hat{x}$.

Performance Distribution of $\widehat{w}^{unif}$

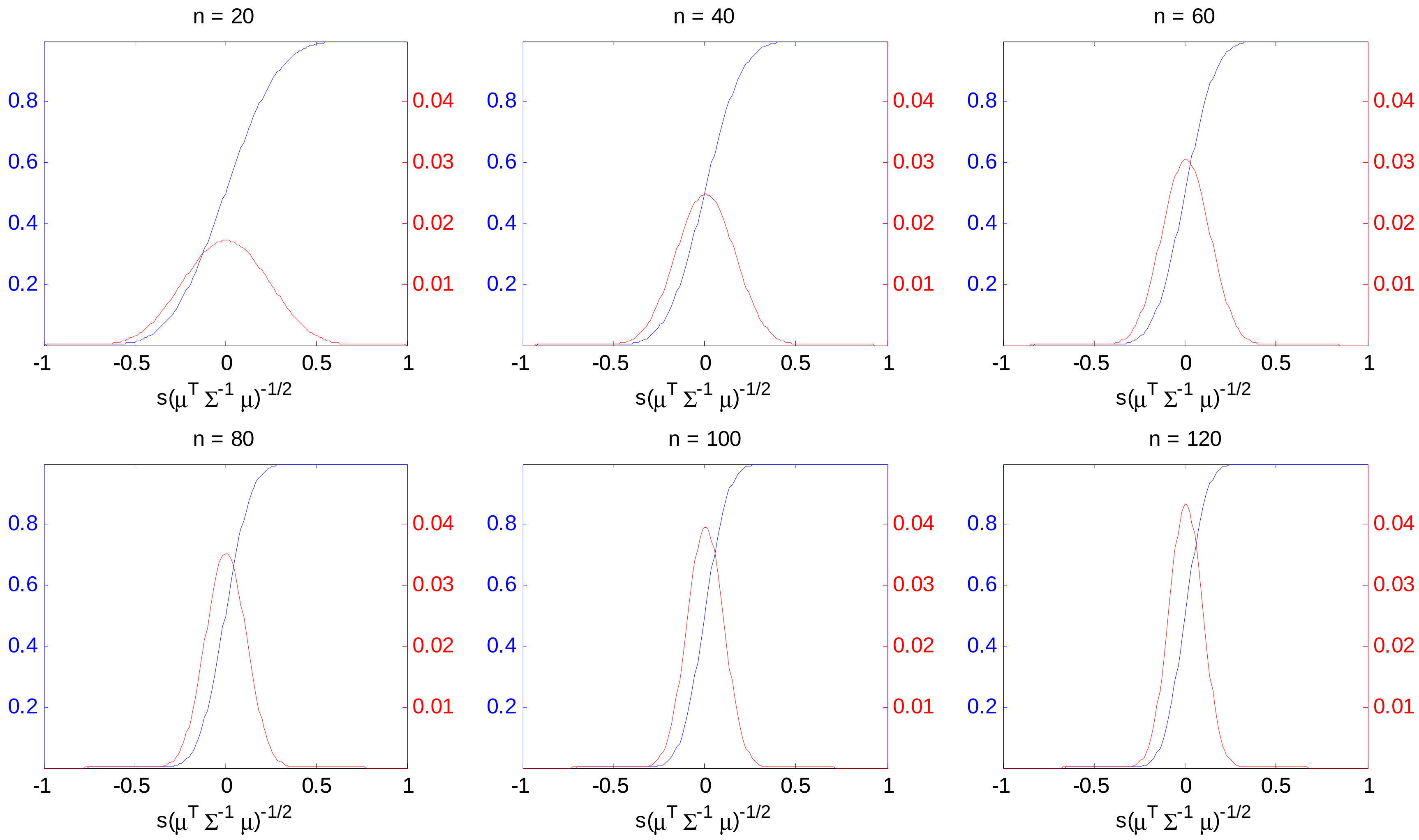
Theorem 1 Let $\mathbb{P}_n^{(\mu,\Sigma)}(s)$ be the probability for the uniformly distributed random portfolio $\widehat{w}^{unif}$ in an asset universe with true parameters (μ, Σ) to have the Sharpe ratio equivalent to or worse than

$$s \in \left[-\sqrt{\mu^T \Sigma^{-1} \mu}, \sqrt{\mu^T \Sigma^{-1} \mu}\right].$$

Then, for the regularized incomplete beta function $I_x(z, w)$,

$$\mathbb{P}_n^{(\mu,\Sigma)}(s) := \mathbb{P}\left(SR^0\left(\widehat{w}^{unif} \middle| \mu, \Sigma\right) \leq s\right)$$
$$= \begin{cases} 1 - \dfrac{1}{2} I_{\sin^2\left(\arccos\left(\frac{s}{\sqrt{\mu^T \Sigma^{-1} \mu}}\right)\right)}\left(\dfrac{n-1}{2}, \dfrac{1}{2}\right) & if\ s \geq 0 \\ \dfrac{1}{2} I_{\sin^2\left(\arccos\left(\frac{s}{\sqrt{\mu^T \Sigma^{-1} \mu}}\right)\right)}\left(\dfrac{n-1}{2}, \dfrac{1}{2}\right) & else. \end{cases}$$

PDF (red) and CDF (blue) of the Sharpe ratio of $\widehat{w}^{unif}$



Sketch of Proof – Step 1

First, for the true parameters μ and Σ, let $x = \Sigma^{-1}\mu$.

Let $\widehat{w}$ and $\widehat{w}^{ref}$ be the portfolios constructed by a skilled manager where their aggregated estimators are $\hat{x}$ and $\hat{x}^{ref}$, respectively.

And let

$$\begin{aligned} \theta &= \text{angle between } \hat{x} \text{ and } x \\ \theta^{ref} &= \text{angle between } \hat{x}^{ref} \text{ and } x \end{aligned}$$

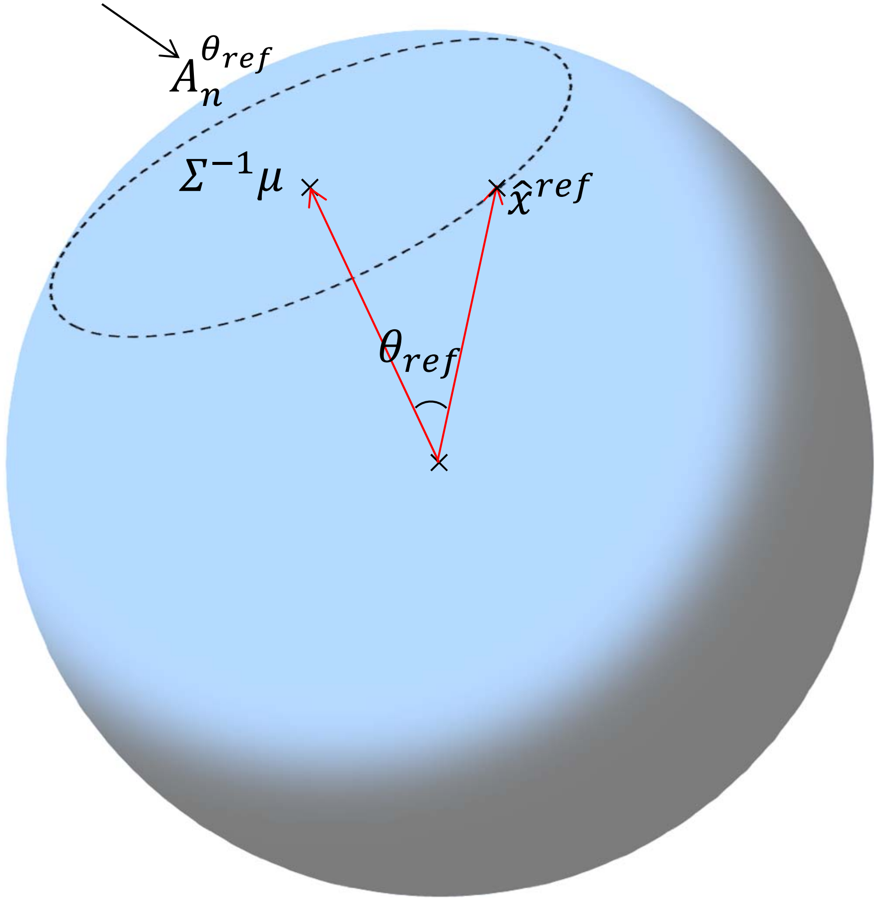
Then, $\widehat{w}$ outperforms $\widehat{w}^{ref}$ in Sharpe ratio if

$$\theta \leq \theta^{ref}.$$

Sketch of Proof – Step 2

Since we have restricted the aggregated estimators $\hat{x}$ and $\hat{x}^{ref}$ to reside on the surface of unit hypersphere,

Hyperspherical cap that $\hat{x}$ can reside within which $\hat{w}$ outperforms $\hat{w}^{ref}$



Sketch of Proof – Step 3

Now, let's replace $\hat{w}$ and $\hat{x}$ with $\hat{w}^{unif}$ and $\hat{x}^{unif}$. Recall that $\hat{x}^{unif}$ is uniformly distributed on the surface of unit hypersphere.

Thus,

$$
\begin{aligned}
&\mathbb{P}\left(\hat{w}^{unif}\ outperforms\ \hat{w}^{ref}\right) \\
&= \mathbb{P}\left(\hat{x}^{unif}\ resides\ on\ A_n^{\theta_{ref}}\right) \\
&= \left(surface\ area\ of\ A_n^{\theta_{ref}}\right) / (surface\ area\ of\ unit\ hypersphere)
\end{aligned}
$$

But We Know Asset Values Grow Over Time…

Thus, we will extend Theorem 1 by allowing the skilled manager to focus more (or less) on the halfspace $\{x_\Sigma \mid \mathbf{1}_\Sigma^T x_\Sigma \geq 0\}$

__Definition__ For $\mathbf{1}_\Sigma = L_\Sigma^T \mathbf{1}$, let H^+ and H^- be two halfspaces separated by the hyperplane $H = \{x_\Sigma \mid \mathbf{1}_\Sigma^T x_\Sigma = 0\}$, namely, $H^+ = \{x_\Sigma \mid \mathbf{1}_\Sigma^T x_\Sigma \geq 0\}$ and $H^- = \{x_\Sigma \mid \mathbf{1}_\Sigma^T x_\Sigma \leq 0\}$. Consider a random manager who chooses $\hat{\mu}$ and $\hat{\Sigma}$ upon constructing her portfolio $\widehat{w}_+^H$ (or $\widehat{w}_-^H$) so that $x_\Sigma = L_\Sigma^T \hat{\Sigma}^{-1} \hat{\mu}$ is uniformly distributed within H^+ (or H^-). Then, $\widehat{w}_+^H$ (or $\widehat{w}_-^H$) is the uniformly distributed random portfolio in the halfspace H^+ (or H^-).

Performance Distribution of $\hat{w}_{+}^{H}$

Proposition Let $\hat{w}^{ref}$ be a reference portfolio with aggregated estimator $\hat{x}^{ref}$. Also, let $\theta_{\mathbf{1}}$ and $\theta_{ref} \in [0,\pi]$ be the angle between $L_{\Sigma}^{T}\mathbf{1}$ and $(L_{\Sigma}^{T})^{-1}\mu$ and the angle between $L_{\Sigma}^{T}\hat{x}^{ref}$ and $(L_{\Sigma}^{T})^{-1}\mu$, respectively. In addition, let

$$\bar{\theta}_{ref} = \begin{cases} \theta_{ref} & if\ \theta_{ref} \in [0,\pi/2] \\ \pi - \theta_{ref} & else \end{cases} \quad and\ \bar{\theta}_{\mathbf{1}} = \begin{cases} \pi/2 - \theta_{\mathbf{1}} & if\ \theta_{\mathbf{1}} \in [0,\pi/2] \\ \theta_{\mathbf{1}} - \pi/2 & else \end{cases}.$$

Furthermore, let

$$A_n(\bar{\theta}_{ref}) = I_{\sin(\bar{\theta}_{ref})^2}\left(\frac{n-1}{2},\frac{1}{2}\right)$$

$$J_n(\bar{\theta}_{\mathbf{1}},\bar{\theta}_{ref}) = \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{1}{2}}\Gamma\left(\frac{n-1}{2}\right)}\int_{\bar{\theta}_{\mathbf{1}}}^{\bar{\theta}_{ref}} \sin(\phi)^{n-2}\, I_{1-\left(\frac{tan(\bar{\theta}_{\mathbf{1}})}{tan(\phi)}\right)^2}\left(\frac{n-2}{2},\frac{1}{2}\right)\mathrm{d}\phi\,.$$

Then,

$$\begin{aligned}\mathbb{P}_{n,H^+}^{(\mu,\Sigma)}(\theta_{ref}) &\coloneqq \mathbb{P}\left(SR_{\hat{w}_{+}^{H}} \geq SR_{\hat{w}^{ref}}\right) \\ &= \left(1-\mathbb{I}_{\left[0,\frac{\pi}{2}\right]}(\theta_{ref})\right) + \left(\mathbb{I}_{\left[0,\frac{\pi}{2}\right]}(\theta_{ref}) + \mathbb{I}_{\left[0,\frac{\pi}{2}\right]}(\theta_{\mathbf{1}}) - 1\right)A_n(\bar{\theta}_{ref}) \\ &+ \left(1-2\mathbb{I}_{\left[0,\frac{\pi}{2}\right]}(\theta_{\mathbf{1}})\right)\mathbb{I}_{\left[0,\frac{\pi}{2}\right]}(\bar{\theta}_{ref}-\bar{\theta}_{\mathbf{1}})J_n(\bar{\theta}_{\mathbf{1}},\bar{\theta}_{ref}).\end{aligned}$$

Uniformly Distributed Random Portfolio with Priors on Asset Growth

Definition A uniformly distributed random portfolio with the asymmetry measure $p \in [0,1]$ with respect to the hyperplane H

$$\widehat{w}_p^H = B_p \widehat{w}_+^H + \left(1 - B_p\right) \widehat{w}_-^H$$

where B_p is a Bernoulli random variable with $\mathbb{E}B_p = p$. In other words, $\widehat{w}_p^H$ is a random portfolio such that x_Σ is chosen from H^+ with probability p, and from H^- with probability $(1-p)$. Moreover, conditioned that x_Σ resides in H^+ (or H^-), it is uniformly distributed within H^+ (or H^-).

As a special case, we have $\widehat{w}_{0.5}^H = \widehat{w}^{unif}$

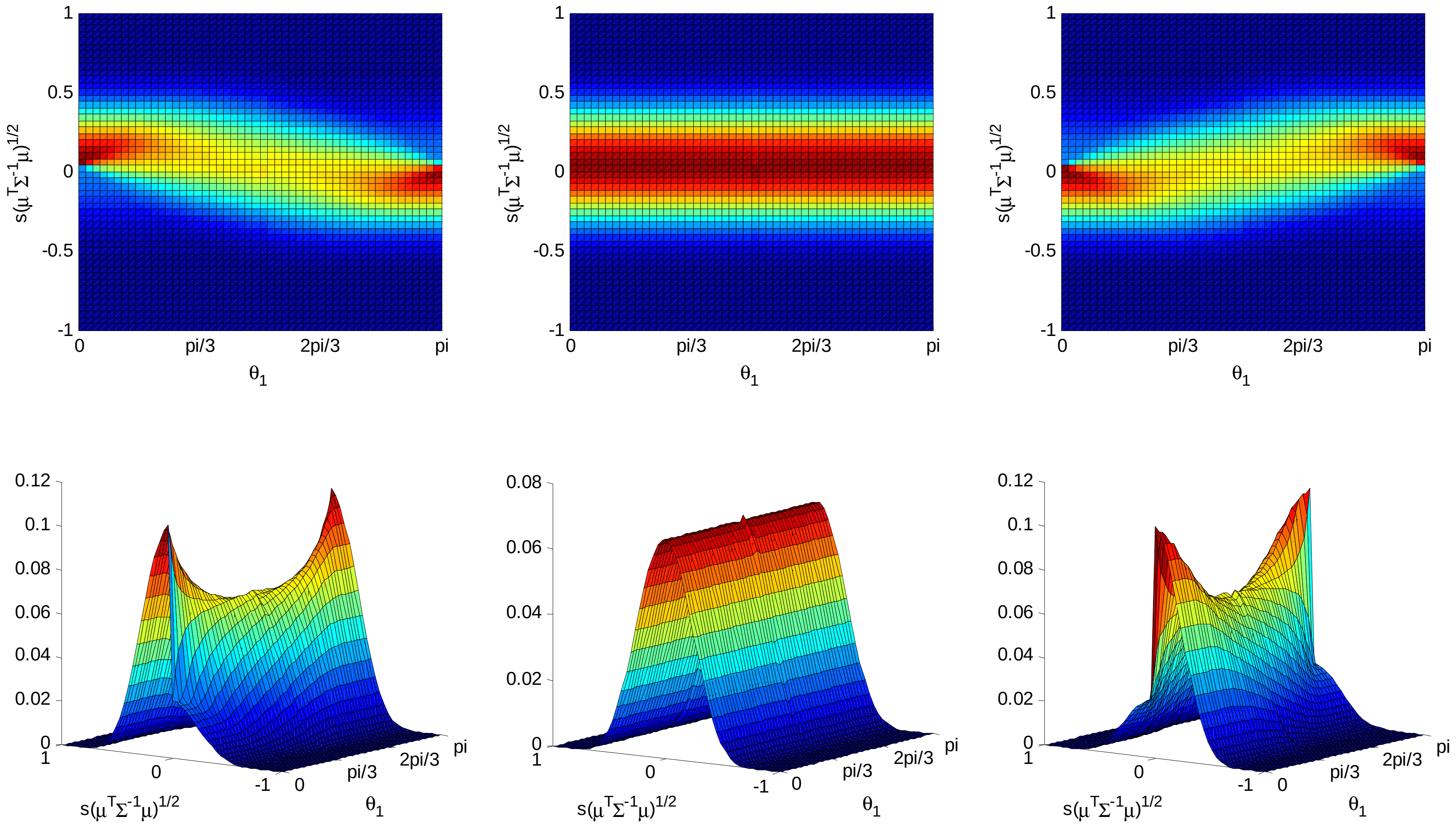
Performance Distribution of $\widehat{w}_p^H$

Theorem 2 For the target Sharpe ratio $s \in \left[-\sqrt{\mu^T \Sigma^{-1} \mu}, \sqrt{\mu^T \Sigma^{-1} \mu}\right]$, the probability for $\widehat{w}_p^H$ to have the Sharpe ratio equivalent to or worse than s

$$\mathbb{P}_{n,H,p}^{(\mu,\Sigma)}(s) \coloneqq \mathbb{P}\left(SR_{\widehat{w}_p^H} \leq s|\mu,\Sigma\right)$$
$$= 1 - \left[p\mathbb{P}_{n,H^+}^{(\mu,\Sigma)}\left(\arccos\left(\frac{s}{\sqrt{\mu^T \Sigma^{-1} \mu}}\right)\right) + (1-p)\mathbb{P}_{n,H^-}^{(\mu,\Sigma)}\left(\arccos\left(\frac{s}{\sqrt{\mu^T \Sigma^{-1} \mu}}\right)\right)\right].$$

Remark H doesn't necessary need to be $\{x_\Sigma \middle| \mathbf{1}_\Sigma^T x_\Sigma = 0\}$. One can choose any arbitrary hyperplane (even the ones that don't contain origin) to conduct similar analysis. In fact, one can do the same analysis with multiple hyperplanes, but the math becomes fairly tedious and ugly.

PDF of $\widehat{w}_p^H$ for $\theta_1 \in [0, \pi]$



(a) $p = 0.8$ (b) $p = 0.5$ (c) $p = 0.2$

Statistical Tests

Null hypothesis

H_0: 1/n strategy does not outperform $\widehat{w}_p^H$ with 50% or more chance.

Let s be the realized Sharpe ratio of 1/n strategy. Then, under the null hypothesis H_0.

$$\mathbb{P}_{n,H,p}^{(\mu,\Sigma)}(s) \coloneqq \mathbb{P}\left(SR^0\left(\widehat{w}_p^H \middle| \mu, \Sigma\right) \le s\right) \le 0.5.$$

We conduct three tests with different values of p as follows

- Test 1: a small number of broad asset indices – equities, bonds, and commodities
- Test 2: a moderate number of equity industry sector indices – 10 ICB (Industry Classification Benchmark) equity indices
- Test 3: a large number of equities – all equities in the US stock market

Results from Test 1

Data: daily returns of three assets from 1980 to 2011

- US stock market index
- US 10-year treasury bonds
- Goldman-Sachs Commodity Index.

Panel A: Annual rebalancing frequency (32 data points)

p	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
Reject H_0 ($\alpha = 0.01$)?	N	N	N	N	N	N	N	N	N
Reject H_0 ($\alpha = 0.05$)?	N	N	N	N	N	N	N	Y	Y
Reject H_0 ($\alpha = 0.10$)?	N	N	N	N	Y	Y	Y	Y	Y

Panel B: Semi-annual rebalancing frequency (64 data points)

p	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
Reject H_0 ($\alpha = 0.01$)?	N	N	N	N	N	N	N	N	N
Reject H_0 ($\alpha = 0.05$)?	N	N	N	N	N	N	N	Y	Y
Reject H_0 ($\alpha = 0.10$)?	N	N	N	N	N	Y	Y	Y	Y

Panel C: Quarterly rebalancing frequency (128 data points)

p	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
Reject H_0 ($\alpha = 0.01$)?	N	N	N	N	N	N	N	N	N
Reject H_0 ($\alpha = 0.05$)?	N	N	N	N	N	N	N	N	Y
Reject H_0 ($\alpha = 0.10$)?	N	N	N	N	N	N	N	Y	Y

Results from Test 2

Data: daily returns of level-2 ICB sector indices from 1973 to 2011

– basic materials (BMATR), consumer goods (CNSMG), consumer services, (CNSMS), financial services (FINAN), health care (HLTHC), industrials (INDUS), oil and gas (OILGS), technologies (TECNO), telecommunications (TELCM) and utilities (UTILS)

Panel A: Annual rebalancing frequency (39 data points)

p	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
Reject H_0 ($\alpha = 0.01$)?	N	N	N	N	N	N	N	N	Y
Reject H_0 ($\alpha = 0.05$)?	N	N	N	N	N	Y	Y	Y	Y
Reject H_0 ($\alpha = 0.10$)?	N	N	N	Y	Y	Y	Y	Y	Y

Panel B: Semi-annual rebalancing frequency (78 data points)

p	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
Reject H_0 ($\alpha = 0.01$)?	N	N	N	N	N	N	N	N	Y
Reject H_0 ($\alpha = 0.05$)?	N	N	N	N	N	Y	Y	Y	Y
Reject H_0 ($\alpha = 0.10$)?	N	N	Y	Y	Y	Y	Y	Y	Y

Panel C: Quarterly rebalancing frequency (156 data points)

p	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
Reject H_0 ($\alpha = 0.01$)?	N	N	N	N	N	N	N	Y	Y
Reject H_0 ($\alpha = 0.05$)?	N	N	N	N	N	Y	Y	Y	Y
Reject H_0 ($\alpha = 0.10$)?	N	N	Y	Y	Y	Y	Y	Y	Y

Results from Test 3

Data: daily returns of all US stocks from 1987 to 2007

- Stocks listed in NYSE, NASDAQ, and AMEX
- Obtained from Datastream
- Number of stocks ranges from 1,374 to 2,724

Panel A: Annual rebalancing frequency (21 data points)

p	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
Reject H_0 ($\alpha = 0.01$)?	N	N	N	N	N	N	N	Y	Y
Reject H_0 ($\alpha = 0.05$)?	N	N	N	N	Y	Y	Y	Y	Y
Reject H_0 ($\alpha = 0.10$)?	N	Y	Y	Y	Y	Y	Y	Y	Y

Panel B: Semi-annual rebalancing frequency (42 data points)

p	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
Reject H_0 ($\alpha = 0.01$)?	N	N	N	N	N	N	N	Y	Y
Reject H_0 ($\alpha = 0.05$)?	N	N	N	N	Y	Y	Y	Y	Y
Reject H_0 ($\alpha = 0.10$)?	N	Y	Y	Y	Y	Y	Y	Y	Y

Panel C: Quarterly rebalancing frequency (84 data points)

p	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
Reject H_0 ($\alpha = 0.01$)?	N	N	N	N	N	N	N	Y	Y
Reject H_0 ($\alpha = 0.05$)?	N	N	N	N	Y	Y	Y	Y	Y
Reject H_0 ($\alpha = 0.10$)?	N	N	Y	Y	Y	Y	Y	Y	Y

Conclusions

- Certainly, 1/n is not bad.

- But at the same time, 1/n doesn't seem to be a very good performer, at least for the data sets we used. For all three cases,
  - We cannot reject the null hypothesis that 1/n is not better than $\hat{w}_{0.9}^{H}$.
  - 1/n is only marginally better than $\hat{w}_{0.5}^{H} = \hat{w}^{unif}$.
  - $\hat{w}_{0.1}^{H}$ does seem to be worse than 1/n.

- From the modeling perspective, the uniformly distributed random portfolio can be employed to evaluate the performance of investment strategies.

Closing Remark: Why $\hat{x}_\Sigma$ Uniformly Distributed, Not $\hat{x}$?

- If $\hat{x} = \hat{\Sigma}^{-1}\hat{\mu}$ is uniformly distributed, instead of $\hat{x}_\Sigma$, then

$$\mathbb{P}\left(SR_{\hat{w}^{unif}} \geq SR_{\hat{w}^{ref}}\right)$$
$$= (surface\ area\ of\ ellipsoidal\ cap)/(surface\ area\ of\ ellipsoid)$$

- Consequently, $\mathbb{P}_{n,H,p}^{(\mu,\Sigma)}(s)$ in Theorem 2 is not anymore a single dimensional integration of the regularized incomplete beta function.

- It is rather given by an $(n-1)$-dimensional integration, which could be quite difficult to compute numerically for the large values of n, if not impossible

- Making $\hat{x}_\Sigma$ to be uniformly distributed instead of $\hat{x}$ allows us to have a closed form expression for the distribution of the Sharpe ratio of $\hat{w}^{unif}$.

Closing Remark: Why $\hat{x}_{\Sigma}$ Uniformly Distributed, Not $\hat{x}$?

- Two possible remedies exist.

- First approach: approximate ellipsoid with a hypersphere

  => This is the exactly what we have done in this study.

- Second approach
  - Employ the factor-based approach so that the value of n is not too large
  - In fact, this approach allows us to relax the assumption that the true covariance matrix is fully ranked and invertible, which makes the proposed framework even more applicable.