Homogeneous structures, ω-categoricity and amalgamation constructions *

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Outline.

The main purpose of these lectures is to give an exposition of some basic material on homogeneous structures, ω-categorical structures and their automorphism groups. There is nothing new in the the talks: most of what is in the first two sessions will be familiar to anyone who has done any work in the area; the third session on Hrushovski's predimension construction is a bit more specialised.

The plan of the lectures is:

1. Homogeneous structures, Fraïssé's theorem and examples; ω-categoricity, the Ryll-Nardzewski Theorem, more examples.

2. Automorphism groups as topological groups; imaginaries and biinterpretability for ω-categorical structures.

3. Generalizations of the Fraïssé construction. Hrushovski's predimension construction and amalgamation. Using the Hrushovski construction to produce ω-categorical structures.

These notes are rather casual about attribution of results: the references indicated below provide more background information and detail. In many places, there are strong similarities to the notes of Macpherson [23].

General background on model theory can be found in standard texts such as [25], [13] or [27]. A short appendix (Section 4, essentially reproduced from [11]) covers some of the basics. Introductory material on ω-categoricity can be found in the introduction to [19] (and many other places), and the book [6] focuses on the connections with permutation groups. The paper [10] gives a survey of constructions of ω-categorical structures, including the examples described here. Macpherson's MALOA lectures [23], and the paper [24], give an extensive survey of work on homogeneous structures and their automorphism groups. The introduction to [7] surveys work on classification of homogeneous structures.

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1 Homogeneous and ω-categorical structures

1.1 Notation and terminology

Throughout L will denote a first-order language (usually countable). This will always include a symbol for equality, which all structures will interpret as true equality. We will not distinguish between an L-structure M and its domain. If $\bar{a} = (a_1, \ldots, a_n)$ is a finite tuple of elements of M, we might write $\bar{a} \in M$ (rather than $\bar{a} \in M^n$).

If M is an L-structure then $\mathrm{Aut}(M)$ is the automorphism group of M. We think of this as acting on the left: so if $g \in \mathrm{Aut}(M)$ and $a \in M$ then we write ga or $g(a)$ (rather than ag or a^g). We also think of $\mathrm{Aut}(M)$ as acting on M^n via the diagonal action: $g\bar{a} = (ga_1, \ldots, ga_n)$.

If $B \subseteq M$ the pointwise stabilizer of B in $\mathrm{Aut}(M)$ is

$$\mathrm{Aut}(M/B) = \{g \in \mathrm{Aut}(M) : gb = b \ \forall b \in B\}.$$

If G is a group acting on a set X and $a \in X$ then the G-orbit which contains a is $\{ga : g \in G\} \subseteq X$. This is the equivalence class containing a for the equivanece relation $a \sim b \Leftrightarrow (\exists g \in G)(ga = b)$. If there is a unique G-orbit on X we say that G is transitive on X. If $a \in X$, then let $G_a = \{g \in G : ga = a\}$ be the stabilizer of a in G. There is a canonical bijection, respecting the G-action, betweeen the set of left cosets of G_a in G and the G-orbit containing a, given by

$$gG_a \mapsto ga.$$

In particular, the index of G_a in G is the cardinality of the G-orbit which contains a. (This is sometimes called the *Orbit- Stabilizer Theorem.*)

Exercise: Show that if M is countable and B is a finite subset of M, then $\mathrm{Aut}(M/B)$ is a subgroup of countable index in $\mathrm{Aut}(M)$.

1.2 Amalgamation classes and homogeneous structures

We are interested in (countable) structures with 'large' automorphism groups. One possible interpretation of this is the following.

Definition 1.1. An L-structure M is *homogeneous* if isomorphisms beteween finitely generated substructures extend to automorphisms of M, that is: if $A_1, A_2 \subseteq M$ are f.g. substructures and $f : A_1 \to A_2$ is an isomorphism, then there exists $g \in \mathrm{Aut}(M)$ such that $g|A_1 = f$.

Remarks 1.2. 1. (Warning) Suppose M is any L-structure. For each $n \in \mathbb{N}$ and each $\mathrm{Aut}(M)$-orbit S on M^n, introduce a new n-ary relation symbol R_S into the language. Call the resulting language L^+. We regard M as an L^+-structure M^+ by interpreting a new relation symbol R_S as the orbit S. Then M^+ is a homogeneous L^+-structure and the automorphism group of M^+ is still $\mathrm{Aut}(M)$.

2. If L is a finite relational language, then there are only finitely many isomorphism types of L-structure of any finite size. So if M is a homogeneous L-structure, then $\mathrm{Aut}(M)$ has finitely many orbits on M^n for all $n \in \mathbb{N}$.

3. Let L consist of a single 2-ary relation symbol and consider the L-structure $M = (\mathbb{Q}; \leq)$, the rationals with their usual ordering. This is a homogeneous L-structure (one way to see this: use piecewise linear automorphisms).

Definition 1.3. A non-empty class $\mathcal{A}$ of finitely generated L-structures is a (Fraïssé) *amalgamation class* if:

1. (IP) $\mathcal{A}$ is closed under isomorphisms;

2. (Hereditary Property, HP) $\mathcal{A}$ is closed under f.g. substructures;

3. (Joint Embedding Property, JEP) if $A_1, A_2 \in \mathcal{A}$ there is $C \in \mathcal{A}$ and embeddings $f_i : A_i \to C$ $(i = 1, 2)$;

4. (Amalgamation Property, AP) if $A_0, A_1, A_2 \in \mathcal{A}$ and $f_i : A_0 \to A_i$ are embeddings, there is $B \in \mathcal{A}$ and embeddings $g_i : A_i \to B$ with $g_1 \circ f_1 = g_2 \circ f_2$.

Remarks 1.4. 1. Note that if $\emptyset \in \mathcal{A}$ then JEP follows from AP.

2. As an example, let L consist of a 2-ary relation symbol R and $\mathcal{A}$ the class of all finite graphs (considered as vertex sets with R interpreted as adjacency). This is an amalgamation class. To verify AP, regard f_1, f_2 as inclusions and let B be the disjoint union of A_1 and A_2 over A_0 with edges $R^{A_1} \cup R^{A_2}$. Take g_1, g_2 to be the natural inclusions. We refer to B as the *free amalgam* of A_1, A_2 over A_0 (and sometimes denote it by $A_1 \coprod_{A_0} A_2$).

Definition 1.5. Suppose M is an L-structure. The *age* of M, $\mathrm{Age}(M)$ is the class of structures isomorphic to some f.g. substructure of M.

Theorem 1.6. *(Fraïssé's Theorem)*

1. *If M is a homogeneous L-structure, then* $\mathrm{Age}(M)$ *is an amalgamation class.*

2. *Conversely, if $\mathcal{A}$ is an amalgamation class of countable L-structures, with countably many isomorphism types, then there is a countable homogeneous L-structure M with $\mathcal{A} = \mathrm{Age}(M)$.*

3. *Suppose $\mathcal{A}$ is as in (2) and M is a countable homogeneous L-structure with age $\mathcal{A}$. Then M has the property that if $A \subseteq M$ is f.g. and $f : A \to B$ is an embedding with $B \in \mathcal{A}$, then there is an embedding $g : B \to M$ with $g(f(a)) = a$ for all $a \in A$. This property determines M up to isomorphism amongst countable structures with age $\mathcal{A}$.*

Definition 1.7. In the above, the structure M is determined up to isomorphism by $\mathcal{A}$ and is referred to as the *Fraïssé limit*, or *generic structure* of $\mathcal{A}$. The property in (3) is sometimes called the *Extension Property*

Examples 1.8. We give some examples of amalgamation classes and homogeneous structures. In each case, the language is the 'natural' language for the structures.

1. The class of all finite graphs is an amalgamation class. The Fraïssé limit is the *random graph*.
2. If $n \geq 3$, let K_n denote the complete graph on n vertices. Consider the class of all finite graphs which do not embed K_n. This is an amalgamation class (free amalgamation gives AP) and the Fraïssé limit is sometimes called the generic K_n-free graph.
3. As with graphs, the class of all finite directed graphs is an amalgamation class. We can use a similar idea to (2) to construct continuum many homogeneous directed graphs. Recall that a tournament is a directed graph with the property that for every two vertices a, b, one of $(a, b), (b, a)$ is a directed edge. There is an infinite set $\mathcal{S}$ of finite tournaments with the property that if A, B are distinct elements of $\mathcal{S}$ then A does not embed in B. If $\mathcal{T}$ is a subset of $\mathcal{S}$, consider the class of finite directed graphs which do not embed any member of $\mathcal{T}$. This is an amalgamation class (use free alamgamation); call the Fraïssé limit $H(\mathcal{T})$. It is easy to see that the elements of $\mathcal{S}$ which are in $\mathrm{Age}(H(\mathcal{T}))$ are the elements of $\mathcal{S} \setminus \mathcal{T}$. So the $H(\mathcal{T})$ are all non-isomorphic. These are called the *Henson digraphs*.
4. The class of all finite linear orders is an amalgamation class (but we cannot use free amalgamation). The Fraïssé limit is isomorphic to $(\mathbb{Q}; \leq)$.
5. The class of all finite partial orders is an amalgamation class.
6. The class of all finite groups is an amalgamation class. The generic structure is Philip Hall's universal locally finite group.
7. (Cherlin, [7]) Let L consist of 3 binary relation symbols G, R, B and consider the class of finite L-structures where R, G, B are symmetric, and for every pair of elements, exactly one of R, G, B holds. So these are complete graphs where each edge is coloured R, G or B. Consider the subclass of structures which omit the triangles:

$$RBB, GGB, BBB.$$

   This is an amalgamation class. Amalgamation can be performed using only R, G edges, but a single edge colour will not suffice.

Proof of Theorem 1.6: We sketch a few details of the proof of Fraïssé 's Theorem.

1. Suppose M is a homogeneous L-structure. We show that $\mathrm{Age}(M)$ is an amalgamation class and that M has the Extension Property in 1.6(3). It is easy to see that $\mathrm{Age}(M)$ has IP, HP and JEP, so we verify AP.

Use the notation in the Definition. Without loss we can assume that $A_1, A_2 \subseteq M$ and $f_1 : A_0 \to A_1$ is the inclusion map. Thus $f_2 : A_0 \to A_2$ is an embedding between subsets of M. Call the image B_0. So we have (from f_2) an isomorphism $A_0 \to B_0$. By homogeneity this extends to an automorphism h of M. Let B be the substructure generated by $A_1 \cup h^{-1}(A_2)$, let $g_1 : A_1 \to C$ be inclusion and $g_2 : A_2 \to C$ be $h^{-1}|A_2$. If $a \in A_0$ then $g_2(f_2(a)) = a = g_1(f_1(a))$, as required.

The proof of EP is similar. There is some embedding $k : B \to M$. Let $A' = k(A)$. Then k gives an isomorphism $A \to A'$, which extends to an automorphism h of M. Let $g = h^{-1} \circ k : B \to M$. Then $g(a) = a$ for all $a \in A$, as required.

2. Suppose M, M' are countable L-structures with age $\mathcal{A}$ and which have EP. Suppose $A \subseteq M$ and $A' \subseteq M'$ are f.g. substructures and $k : A \to A'$ is an isomorphism. Using a back-and-forth argument, we can show that that k extends to an isomorphism between M and M'. This shows that any two countable structures with EP are isomorphic, and that any countable structure with EP is homogeneous.

3. To finish the proof, it therefore remains to show that if $\mathcal{A}$ is an amalgamation class of countable L-structures with countably many isomorphism types, then there is a countable structure M with age $\mathcal{A}$ which has EP.

Note first that if $A, B \in \mathcal{A}$, then there are countably many embeddings $A \to B$. We build M inductively as the union of a chain of structures in $\mathcal{A}$:

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots.$$

When doing this we ensure that:

- if $C \in \mathcal{A}$, then C embeds into some A_i;
- if A is a f.g. substructure of A_i and $f : A \to B \in \mathcal{A}$, then there is $j > i$ such that there is an embedding $g : B \to A_j$ with $g(f(a)) = a$ for all $a \in A$.

Note that there are countably many tasks to perform here; as we have a countable number of steps at our disposal, it therefore suffices to show that any one of these can be performed. A task of the first form can be performed using JEP. For the second, suppose the construction has reached stage $k > i$. At the next stage we can take A_{k+1} which solves the amalgamation problem $A \to A_k$ (inclusion), $f : A \to B$. Specifically, using AP we obtain $h : A_k \to A_{k+1}$ (which can be taken as inclusion), and $g : B \to A_{k+1}$ with $g(f(a)) = h(a) = a$ for all $a \in A$, as required. □

1.3 ω-categoricity

Suppose L is a first-order language. By the cardinality of L we mean the cardinality of the set of L-formulas. We shall usually work with countable languages. Recall that a *closed* L-formula (or *L-sentence*) is an L-formula without free variables. If M is an L-structure then a closed formula σ makes an assertion about M which is either true or false (written $M \models \sigma$ and $M \not\models \sigma$ respectively). The *theory* of M, denoted by $Th(M)$, is the set of closed formulas which are true in M.

Of course, if M is finite, then $Th(M)$ determines M (up to isomorphism). However, if M is infinite, then, by the Löwenheim - Skolem Theorems, $Th(M)$ will have at least one model of every cardinality greater than or equal to the cardinality of L.

Definition 1.9. Suppose L is a countable language and M is a countably infinite L-structure. We say that M (or $Th(M)$) is *ω-categorical* if every countable model of $Th(M)$ is isomorphic to M.

Proposition 1.10. *Suppose L is countable relational language and M is a countably infinite homogeneous L-structure. Suppose further that for each $n \in \mathbb{N}$, there are finitely many isomorphism types of substructures of M with n elements. Then M is ω-categorical.*

Proof. This will follow from the Ryll-Nardzewski Theorem below, but it's perhaps instructive to give a direct proof. For simplicity we do this when the language has only finitely many relation symbols.

First, note that $Th(M)$ specifies the age of M: for each n we have a closed formula (of the form $(\forall x_1 \ldots x_n) \ldots$) specifying what the isomorphism type of an n-set can be; moreover we have formulas (of the form $(\exists x_1 \ldots x_n) \ldots$) saying that all these are represented.

Second, note that $Th(M)$ also specifies the Extension Property. For each $A \subseteq B \in \mathrm{Age}(M)$ we have in $Th(M)$ the closed formula:

$$(\forall \bar{x})(\exists \bar{y})(\Delta_A(\bar{x}) \to \Delta_{A,B}(\bar{x}, \bar{y}))$$

where $\bar{x}$ is a tuple of variables of length $|A|$ and $\Delta_A(\bar{x})$ is the basic diagram of A, indicating the isomorphism type of A; similarly $\bar{x}\bar{y}$ has length $|B|$ and $\Delta_{A,B}(\bar{x}\bar{y})$ is the basic diagram of B where the variables $\bar{x}$ pick out the substructure A.

It follows that if M' is a model of $Th(M)$ then M' has the same age as M and has the extension property. So if M' is countable, then it is isomorphic to M. $\square$

We recall some model-theoretic terminlogy. Suppose M is an L-structure and $\theta(x_1, \ldots, x_n)$ an L-formula with free variables amongst $x_1, \ldots, x_n$. Let

$$\theta[M] = \{(a_1, \ldots, a_n) \in M^n : M \models \theta(a_1, \ldots, a_n)\}.$$

This is called a *$\emptyset$-definable* subset of M^n. We say that formulas $\theta(\bar{x})$ and $\psi(\bar{x})$ are *equivalent* (modulo $Th(M)$) if they define the same subset of M^n. Equivalently $(\forall \bar{x})(\theta(\bar{x}) \leftrightarrow \psi(\bar{x})) \in Th(M)$.

More generally if $C \subseteq M$, a C-definable subset of M^n is of the form

$$\eta[M, \bar{c}] = \{\bar{a} \in M^n : M \models \eta(\bar{a}, \bar{c})\}$$

for some L-formula $\eta(\bar{x}, \bar{y})$ and tuple $\bar{c}$ of elements of C. The $\bar{c}$ here are called *parameters*, and $\eta(\bar{x}, \bar{c})$ is a formula with parameters from C.

Suppose $\bar{a}$ is an n-tuple of elements of M and $C \subseteq M$. The *type* of $\bar{a}$ over C (in M), written $\mathrm{tp}^M(\bar{a}/C)$ is the set of formulas $\eta(\bar{x}, \bar{c})$ with parameters from C such that $M \models \eta(\bar{a}, \bar{c})$. Note that if $g \in \mathrm{Aut}(M/C)$ then $\mathrm{tp}^M(g\bar{a}/C) = \mathrm{tp}^M(\bar{a}/C)$.

Theorem 1.11. *(Ryll-Nardzewski, Svenonius, Engeler) Suppose L is a countable first-order language and M a countably infinite L-structure. Then the following are equivalent:*

1. *M is ω-categorical;*

2. $\mathrm{Aut}(M)$ *has finitely many orbits on M^n for all $n \in \mathbb{N}$;*

3. *For each $n \in \mathbb{N}$, every n-type of $Th(M)$ is principal;*

4. *For each $n \in \mathbb{N}$ there are only finitely many equivalence classes of L-formulas with n free variables (modulo $Th(M)$).*

Remarks 1.12. 1. For a proof, see for example ([25], 4.4.1). One way to organise the proof is

$$(2) \Rightarrow (4) \Rightarrow (3) \Rightarrow (1) \Rightarrow (3) \Rightarrow (2).$$

All but one of these are either straightforward or an application of compactness. The exception is (1) $\Rightarrow$ (3) which uses the Omitting Types Theorem.

2. A type is *principal* if it contains a formula which implies all of the other formulas in it.

3. It is clear that (2) $\Rightarrow$ (1) gives Proposition 1.10.

4. We say that a group G acting on a set X is *oligomorphic* if G has finitely many orbits on X^n for all $n \in \mathbb{N}$.

Example 1.13. We give an example of how amalgamation constructions can sometimes be used to produce ω-categorical structures (and oligomorphic groups) with prescribed properties.

Suppose $(k_n : n \in \mathbb{N})$ is a given sequence of natural numbers. We construct an ω-categorical structure M such that for every $n \in \mathbb{N}$, the number of orbits of $\mathrm{Aut}(M)$ on M^n is at least k_n. Consider a language L which has k_n n-ary relation symbols for each n. Let $\mathcal{A}$ consist of finite L-structures C such that for each relation symbol R, if $C \models R(c_1, \ldots, c_n)$, then $c_1, \ldots, c_n$ are distinct. This is an amalgamation class (use free amalgamation) and for each n there are only finitely many, but at least k_n, isomorphism types of structures of size n in $\mathcal{A}$. So the Fraïssé limit M is ω-categorical and has the required property.

Corollary 1.14. *Suppose M is ω-categorical.*

1. *Two n-tuples are in the same* $\mathrm{Aut}(M)$*-orbit iff they have the same type over $\emptyset$ in M.*

2. *The $\emptyset$-definable subsets of M^n are precisely the* $\mathrm{Aut}(M)$*-invariant subsets of M^n, that is, unions of* $\mathrm{Aut}(M)$*-orbits on M^n.*

3. *If $C \subseteq M$ is finite, then the C-definable subsets of M^n are precisely the* $\mathrm{Aut}(M/C)$*-invariant sets.*

Proof. (1) The direction $\Rightarrow$ is true in general; the other direction is part of the proof of (3) $\Rightarrow$ (2) in the above.

(2) It is clear that a $\emptyset$-definable subset of M^n is $\mathrm{Aut}(M)$-invariant, so is a union of $\mathrm{Aut}(M)$-orbits on M^n. It follows that it is enough to show that each such $\mathrm{Aut}(M)$-orbit X is definable. But this follows from (1) and the fact that types are principal.

(3) Expand the language to the language $L(C)$ by adding new constants for the elements of C. Regard M as an $L(C)$-structure $(M; C)$ in the obvious way and note that a C-definable subset of M^n is the same thing as a $\emptyset$-definable subset of the $L(C)$-structure $(M; C)$. The automorphism group of the latter is $\mathrm{Aut}(M/C)$ and as C is finite, this has finitely many orbits on n-tuples for all n. So $(M; C)$ is ω-categorical and (3) follows from (2). □

The following characterization of homogeneous structures amongst the ω-categorical structures is worth noting. Recall that an L-structure M (or its theory $Th(M)$) is said to have *quantifier elimination* (QE) if for every $n \geq 1$, every L-formula $\theta(x_1, \ldots, x_n)$ is equivalent (modulo $Th(M)$) to a quantifier-free formula $\eta(x_1, \ldots, x_n)$.

Theorem 1.15. *Suppose M is an ω-categorical L-structure. Then $Th(M)$ has quantifier elimination iff M is homogeneous.*

Proof. ($\Rightarrow$:) Suppose A_1, A_2 are f.g. substructures of M and $f : A_1 \to A_2$ is an isomorphism. So by ω-categoricity, A_1, A_2 are finite. Let $\bar{a}_1$ enumerate A_1 and $\bar{a}_2 = f(a_2)$. Then $\bar{a}_1$, $\bar{a}_2$ satisfy the same quantifier-free formulas in M. By QE, it follows that $\mathrm{tp}^M(\bar{a}_1/\emptyset) = \mathrm{tp}^M(\bar{a}_2/\emptyset)$. By Corollary 1.14(1), there is $g \in \mathrm{Aut}(M)$ with $g\bar{a}_1 = \bar{a}_2$. Thus g extends f, as required.

($\Leftarrow$:) If $\bar{a}, \bar{a}'$ are tuples in M with the same quantifier free type, then $\bar{a} \mapsto \bar{a}'$ extends to an isomorphism $f : A \to A'$ between the substructures generated by $\bar{a}, \bar{a}'$. By homogeneity, there is an automorphism of M extending f and so $\bar{a}, \bar{a}'$ have the same type over $\emptyset$ in M. So quantifier-free types determine types (over $\emptyset$) in M; as all types are principal it follows that every formula is equivalent to a quantifier free formula (in M). □

We conclude this subsection with some comments on *algebraic closure.*

Definition 1.16. Suppose M is an L-structure and $A \subseteq M$. The *algebraic closure* $\mathrm{acl}(A)$ of A in M is the union of the finite A-definable subsets of M. In general, $\mathrm{acl}(A)$ contains the substructure generated by A and acl is a closure operation on M.

Lemma 1.17. *Suppose M is ω-categorical. Then* acl *is a uniformly locally finite closure operation on M: there is a function $\alpha : \mathbb{N} \to \mathbb{N}$ such that if $A \subseteq M$ is finite, then $|\mathrm{acl}(A)| \leq \alpha(|A|)$.*

Proof. By the previous corollary, $\mathrm{acl}(M)$ is the union of the finite $Aut(M/A)$-orbits on M, so is finite. Note that if $g \in \mathrm{Aut}(M)$ then $\mathrm{acl}(gA) = g\mathrm{acl}(A)$. So as there are only finitely many orbits on finite sets of any given size, there is a uniform bound on the size of the algebraic closures of these sets. □

In particular, if M is ω-categorical there is a uniform bound on the size of n-generator substructures, for all $n \in \mathbb{N}$ (of course, if L is a relational language, this is not saying very much).

We say that algebraic closure in M is *trivial* if $\mathrm{acl}(A) = A$ for all finite $A \subseteq M$. If M is homogeneous, this can be expressed as a condition on its age:

Lemma 1.18. *Suppose L is a countable relational language and M is a homogeneous L-structure which is ω-categorical. Then algebraic closure in M is trivial iff $\mathcal{A} = \mathrm{Age}(M)$ satisfies:*

(Strong Amalgamation Property) if $A_0, A_1, A_2 \in \mathcal{A}$ and $f_i : A_0 \to A_i$ are embeddings, there is $B \in \mathcal{A}$ and embeddings $g_i : A_i \to B$ with $g_1 \circ f_1 = g_2 \circ f_2$ and $g_1(A_1) \cap g_2(A_2) = g_1(f_1(A_0))$.

Proof. First, suppose $\mathcal{A}$ has the strong amalgamation property. Let $B \subseteq M$ be finite and $c \notin M$. We have to show that c is in an infinite $\mathrm{Aut}(M/B)$-orbit, so we show that for every $n \in \mathbb{N}$ there are at least n elements in this orbit. Let C be the substructure $B \cup \{c\}$. By strong amalgamation there is a structure D in $\mathcal{A}$ which consists of n distinct copies of C amalgamated over B; so $D = B \cup \{c_1, \ldots, c_n\}$. We can assume $C \subseteq D$ (say $c = c_1$) and using the Extension Property of M, we can assume that $D \subseteq M$. Then the $B \cup \{c_i\}$ are isomorphic (over B), so by homogeneity, the c_i are in the same $\mathrm{Aut}(M/B)$-orbit.

Conversely, suppose algebraic closure is trivial in M. We modify the proof of AP given in Theorem 1.6. Use the notation in the Definition. Without loss we can assume that $A_1, A_2 \subseteq M$ and $f_1 : A_0 \to A_1$ is the inclusion map. Thus $f_2 : A_0 \to A_2$ is an embedding between subsets of M. Call the image B_0. So we have (from f_2) an isomorphism $A_0 \to B_0$. By homogeneity this extends to an automorphism h of M. So $h^{-1}(A_2) \supseteq A_0$. By Neumann's lemma (after applying an element of $\mathrm{Aut}(M/A_0)$) we can assume that $h^{-1}(A_2) \cap A_1 = A_0$. Let $B = A_1 \cup h^{-1}(A_2)$, let $g_1 : A_1 \to C$ be inclusion and $g_2 : A_2 \to C$ be $h^{-1}|A_2$. If $a \in A_0$ then $g_2(f_2(a)) = a = g_1(f_1(a))$, as required. □

The proof made use of the following (see Corollary 4.2.2 of [13] for a proof):

Theorem 1.19. *(Neumann's Lemma) Suppose G is a group acting on a set X and all G-orbits on X are infinite. Suppose B, C are finite subsets of X. Then there is some $g \in G$ with $B \cap gC = \emptyset$.*

Examples 1.20. We list some examples of 'natural' ω-categorical structures. The first three have trivial algebraic closure, the rest, non-trivial.

1. A pure set $(M; =)$. So the language just has equality; the automorphism group is the full symmetric group $\mathrm{Sym}(M)$.

2. A countable structure $(M; E)$ with an equivalence relation E which has infinitely many classes, all of which are infinite.

3. The countable, dense linear ordering without endpoints $(\mathbb{Q}; \leq)$.

4. The countable atomless boolean algebra $(\mathbb{B}; 0, 1, \wedge, \vee, \neg)$.

5. A vector space $V(\aleph_0, q)$ of dimension $\aleph_0$ over a finite field $\mathbb{F}_q$ with q elements. Note that the usual language for vector spaces over a field K consists of $+, -, 0, \lambda_a (a \in K)$ where λ_a is a function symbol for scalar multiplication by a.

6. Any countable abelian group of finite exponent.

7. A classical (symplectic, orthgonal or Hermitian) space $(V(\aleph_0, q) : +, -, 0, ...)$ over a finite field, where ... consists of the extra structure, such as the bilinear or quadratic form.

Remarks 1.21. Other ways of constructing ω-categorical structures include interpretation (see the next section) and boolean powers. The survey [10] discusses the latter in detail.

2 Automorphism groups and imaginaries

In this section we will explain the model-theoretic notion of a structure being interpreted in another structure and the related concept of imaginary elements. For an ω-categorical structure, there is a nice way of viewing imaginary elements in terms of a topology on the automorphism group and we will explain this, leading to a result of Ahlbrandt and Ziegler. We will not have time to discuss the many other applications and questions around the topology of automorphism groups. The surveys by Macpherson [23] and Kechris [20] are good references for these.

2.1 Automorphism groups as topological groups

If X is any non-empty set, the symmetric group $G = \mathrm{Sym}(X)$ is the group of all permutations of X. We regard this as a topological group with open sets being unions of cosets of pointwise stabilizers of finite sets. In other words, the basic open sets are of the form $gG_{(A)}$ for $A \subseteq_{fin} X$ and $g \in G$. Note here that

$$G_{(A)} = \{h \in G : ha = a \forall a \in A\}$$

so

$$gG_{(A)} = \{h \in G : h|A = g|A\}.$$

Note also that each of these basic open sets is also closed (the complement is the union of the other cosets).

If X is countable (say $X = \mathbb{N}$), this is separable and complete metrizable. To see the latter, consider d given by, for $g_1 \neq g_2$,

$$d(g_1, g_2) = 1/n \text{ where } n \text{ is as small as possible with } g_1 n \neq g_2 n.$$

This is a metric for the topology, but it is not complete. To obtain a complete metric, consider

$$d'(g_1, g_2) = d(g_1, g_2) + d(g_1^{-1}, g_2^{-1}).$$

This is a complete metric for the topology. So if X is countable, then $\mathrm{Sym}(X)$ is a *Polish group*.

Lemma 2.1. *Suppose $G \leq \mathrm{Sym}(X)$. Then the closure of G in $\mathrm{Sym}(X)$ is*

$$\bar{G} = \{g \in \mathrm{Sym}(X) : gY = Y \text{ for all } G\text{-orbits } Y \text{ on } X^n, \quad \forall n\}.$$

Proof. First, suppose Y is a subset of X^n. We show that $\{g \in \mathrm{Sym}(X) : gY = Y\}$ is closed. If $gY \not\subseteq Y$ there is some $\bar{y} \in Y$ with $g\bar{y} \notin Y$; so if $g' \in gG_{\bar{y}}$ then $g'Y \not\subseteq Y$. So the complement of $\{g \in \mathrm{Sym}(X) : gY \subseteq Y\}$ is open and therefore this set is closed. Similarly $\{g \in \mathrm{Sym}(X) : gY \supseteq Y\}$ is closed so $\{g \in \mathrm{Sym}(Y) : gY = Y\}$ is closed. Thus the intersection of these over all Y is closed and $\bar{G}$ is therefore contained in this intersection.

It follows that $\{g \in \mathrm{Sym}(X) : gY = Y$ for all G-orbits Y on $X^n, \quad \forall n\}$ is closed and clearly it contains G. So it contains $\bar{G}$.

Finally suppose $g \in \bar{G}$ and Y is a G-orbit on X^n. Then every open neighbourhood of g contains a point of G. So if $\bar{y} \in Y$ there is $h \in G$ with $g\bar{y} = h\bar{y}$. So $g\bar{y} \in Y$. Likewise $g^{-1}\bar{y} \in Y$ so $gY = Y$. □

Corollary 2.2. *A subgroup G of $\mathrm{Sym}(X)$ is closed iff G is the automorphism group of some first-order structure on X.*

Proof. A first-order structure on X is specified by relations and functions on X and the automorphisms are the permutations which preserve these. Note that a permutation preserves a function iff it stabilizes (setwise) its graph. So the automorphsim group is the intersection of the setwise stabilisers of certain subsets of M^n for various n. As in the proof of the lemma, this is a closed subgroup.

Conversely, if $G \leq \mathrm{Sym}(X)$ consider the structure on X which has a relation for each G-orbit on X^n, for each finite n. By the proof of the lemma, the automorphism group of this structure is $\bar{G}$. So if G is closed, the automorphism group is G. □

Remarks 2.3. The structure on X constructed above (with relations the G-orbits on X^n) is sometimes called the *canonical structure* for G on X. Note that if G is oligomorphic (and X is countably infinite) this is an ω-categorical structure.

2.2 Interpretations and imaginaries

Some structures can be built out of others in a definable way. The classical example is the construction of the field of rational numbers from the ring of integers. Another example is algebraic groups over a particular field.

Formalising this leads to the notion of an *interpretation* of one structure in another.

Definition 2.4. Suppose K and L are first-order languages, M a K-structure and N an L-structure. We say that N is *interpretable* in M if for some $n \in \mathbb{N}$ there exist:

1. a $\emptyset$-definable subset D of M^n;

2. a $\emptyset$-definable equivalence relation E on D;

3. a bijection $\gamma : N \to D/E$

such that for every $\emptyset$-definable subset R of N^m the subset of M^{mn} given by

$$\hat{R} = \{(\bar{a}_1, \ldots, \bar{a}_m) \in (M^n)^m : (\gamma^{-1}(\bar{a}_1/E), \ldots, \gamma^{-1}(\bar{a}_m/E)) \in R\}$$

is $\emptyset$-definable in M.

Thus the set N can be identified with a $\emptyset$-definable subset of M^n factored by a $\emptyset$-definable equivalence relation, and with this identification all of the L-structure on N can be derived from the K-definable structure on M. There is a considerable amount of redundancy in the definition: it is only necessary to have $\emptyset$-definability of $\hat{R}$ when R is a distinguished constant or relation, or the graph of a distinguished function.

Example 2.5. Let M be a pure set. Consider the graph $(N; R)$ whose vertices N are subsets of size 2 from M and $R(\{a, b\}, \{c, d\}) \Leftrightarrow |\{a, b\} \cap \{c, d\}| = 1$. Then N can be interpreted in M.

If the equivalence relation E in Definition 2.4 is simply equality on D then we say that N is *definable* in M. If also $D = M$ then we say that N is a *reduct* of M (so N just consists of M with some of its definable structure forgotten). Note that if N is interpretable in M then the interpretation gives us a homomorphism $\mathrm{Aut}(M) \to \mathrm{Aut}(N)$ (with the above notation, if $g \in \mathrm{Aut}(M)$ and $b \in N$ then $g(b) = \gamma^{-1}(g(\gamma(b)))$. If N is a reduct of M then $\mathrm{Aut}(M) \subseteq \mathrm{Aut}(N)$.

Lemma 2.6. *Suppose M is an ω-categorical structure and N is a (countably infinite) structure interpreted in M. Then N is ω-categorical.*

Proof. As noted above, $\mathrm{Aut}(M)$ acts on N via its action on M^n. Then for every $k \in \mathbb{N}$, there are finitely many orbits of $\mathrm{Aut}(M)$ on N^k (because there are finitely many orbits on M^{nk}), so there are finitely many orbits of $\mathrm{Aut}(N)$ on N^k. so the result follows by Theorem 1.11. $\square$

Equivalence classes in D/E as above are referred to as *imaginary* elements of M. Taking the set of all imaginary elements (as D and E range over all $\emptyset$-definable sets and equivalence relations) gives us the set M^{eq}. We wish to regard this as a first-order structure, so we extend the language K of M in a canonical way (to a first-order language K^{eq}), and part of the K^{eq}-theory of M^{eq} describes how the imaginary elements correspond in a $\emptyset$-definable way to the original K-structure M.

More formally, for each such $\emptyset$-definable equivalence relation E on a $\emptyset$-definable subset of M^n we introduce an extra 'sort' S_E of elements consisting of the set of E-classes on D. We add to the language a 1-ary predicate to pick out these elements and a new n-ary function symbol for the natural map $\pi_E : D \to D/E$. The theory contains the formulas saying that π_E is surjective and $(\forall \bar{x}\bar{y})(\pi_E(\bar{x}) = \pi_E(\bar{y}) \leftrightarrow E(\bar{x}, \bar{y}))$. (This is sometimes done only for $\emptyset$-definable equivalence relations on M^n; note that any of our equvalence relations can be extended to such, by making classes of tuples outside D to be singletons. If we do this the definable structure is essentially the same.)

The reader can consult ([13], Section 4.3) for the precise details of how to do all of this. Once we have this concept, it makes sense to extend notions such as 'parameter definable', 'types', 'algebraic closure' etc. to subsets of M^{eq}. Again we refer the reader to ([13]) for further details if the need arises.

Note that a structure N is interpretable in M iff it is definable in M^{eq}.

2.3 Biinterpretability of ω-categorical structures

Recall that if M is an L-structure, then an element e of M^{eq} is an equivalence class of some $\emptyset$-definable equivalence relation E on M^n (for some $n \in \mathbb{N}$). So there is some $\bar{a} \in M^n$ such that $e = \bar{a}/E$. Also recall that the action of $G = \mathrm{Aut}(M)$ on M extends in a natural way to an action on M^{eq}: we have $g(e) = g(\bar{a})/E$ for $g \in G$.

Theorem 2.7. *Suppose M is an ω-categorical structure. The open subgroups of $G = \mathrm{Aut}(M)$ are precisely the subgroups of the form $G_e = \mathrm{Aut}(M/e)$ for $e \in M^{eq}$.*

Proof. First, let $e \in M^{eq}$. So $e = \bar{a}/E$ for some $\emptyset$-definable equivalence relation E on a $\emptyset$-definable subset D of M^n and $\bar{a} \in D$. Then clearly $G_{\bar{a}} \leq G_e$. So G_e is the union of the cosets of $G_{\bar{a}}$ which it contains. Therefore G_e is open in G. (Note that this does not use ω-categoricity.)

Conversely, suppose $H \leq G$ is open. As pointwise stabilizers of finite sets form a base of open neighbourhoods of the identity, there is some n-tuple $\bar{a}$ such that $G_{\bar{a}} \leq H$. Let $D = \{g\bar{a} : g \in G\}$ be the G-orbit which contains $\bar{b}$. Define the 2-ary relation E on D by:

$$E(g_1\bar{a}, g_2\bar{a}) \Leftrightarrow g_2^{-1}g_1 \in H.$$

One checks easily that this is well-defined and is a G-invariant equivalence relation on D. As M is ω-categorical, E is a $\emptyset$-definable subset of M^{2n} (Corollary 1.14(2)).

Let $e = \bar{a}/E$. So $e \in M^{eq}$. By definition

$$g \in G_e \Leftrightarrow g \in H.$$

So $H = G_e$, as required. □

Remarks 2.8. Note that the proof shows that the open subgroups containing $G_{\bar{a}}$ are of the form G_e where e is the equivalence class containing $\bar{a}$ for some G-invariant equivalence relation E on M^n. Such an equivalence class is a union of $G_{\bar{a}}$-orbits on M^n. There are only finitely many such orbits (because G has finitely many orbits on M^{2n}) so there are only finitely many possibilities for the class $\bar{a}/E$. It follows that there are only finitely many subgroups of G which contain $G_{\bar{a}}$; it then follows that there are only countably many open subgroups of G.

Theorem 2.9. *(Ahlbrandt and Ziegler; Coquand) Suppose M, N are ω-categorical structures. Then* $\mathrm{Aut}(M)$ *and* $\mathrm{Aut}(N)$ *are isomorphic as topological groups iff M and N are biinterpretable.*

Before proving this, we explain what is meant by saying that M, N are *biinterpretable.*

Informally, this means that M is interpretable in N (without parameters) and N is interpretable in N *and* the 'composite' interpretations of M in itself and N in itself are definable.

Example 2.10. Consider the followng two structures. The structure M consists of a pure set. The structure N consists of a pure set in which each point is 'doubled'; more formally we have $(N; E)$ where N is countably infinite and E is an equivalence relation with all classes of size 2. Then M is interpretable in N: let $\gamma : M \to N/E$ be any bijection. We also have an interpretation of N in M: let D consist of two copies $M \times \{0\}$, $M \times \{1\}$ of M (this can be arranged[1]). There is a bijection $\delta : N \to D$ such that for all $a \in M$, $\delta\gamma(a) = \{(a, 0), (a, 1)\}$. So each of M, N can be interpreted in the other. As $\mathrm{Aut}(M)$ and $\mathrm{Aut}(N)$ are not isomorphic as groups, the structures are not biinterpretable.

Definition 2.11. Suppose M, N are two structures. If N is interpretable in M then there is an injective map $\alpha : N \to M^{eq}$ (with image contained in finitely many sorts) such that any $\emptyset$-definable subset of N^n maps under α to a $\emptyset$-definable subset of M^{eq}. Note that α extends canonically to a map $\tilde{\alpha} : N^{eq} \to M^{eq}$ which maps $\emptyset$-definable sets to $\emptyset$-definable sets.

Suppose also that M is interpretable in N via $\beta : M \to N^{eq}$. Let $\tilde{\beta} : M^{eq} \to N^{eq}$ be the corresponding extension. We obtain interpretations of M and N in themselves:

$$\tilde{\alpha} \circ \beta : M \to M^{eq} \text{ and } \tilde{\beta} \circ \alpha : N \to N^{eq}.$$

We say that M, N are *biinterpretable* if α and β can be chosen so that these compositions are $\emptyset$-definable maps (in M^{eq} and N^{eq} respectively).

Remarks 2.12. In the example, the composition $\tilde{\gamma} \circ \tilde{\delta}$ gives a map $\sigma : N/E \to N$ with the property that $\sigma(c) \in c$ for all $c \in N/E$. Such a map cannot be $\emptyset$-definable in N.

Proof of Theorem 2.9: Suppose M, N are biinterpretable and α, β are as in the definition. Note that α induces a homomorphism $\alpha^* : \mathrm{Aut}(M) \to \mathrm{Aut}(N)$ given by $\alpha^*(g)(b) = \alpha^{-1}g(\alpha(b))$ (for $g \in \mathrm{Aut}(M)$ and $b \in N$). It is continuous as the preimage of $\mathrm{Aut}(N/b)$ contains $\mathrm{Aut}(M/\alpha(b))$ (for $b \in N$). Likewise, we have a continuous homomorphism $\beta^* : \mathrm{Aut}(N) \to \mathrm{Aut}(M)$.

Now, $\beta^*\alpha^* : \mathrm{Aut}(M) \to \mathrm{Aut}(M)$ and for $g \in G$ and $a \in M$ we have:

$$\beta^*\alpha^*(g)(a) = \beta^{-1}\alpha^*(g)\beta(a) = \beta^{-1}\tilde{\alpha}^{-1}g(\tilde{\alpha}\beta(a)).$$

By the definability (and hence $\mathrm{Aut}(M)$-invariance) of $\tilde{\alpha}\beta$ we have $\tilde{\alpha}\beta(ga) = g(\tilde{\alpha}\beta(a))$, thus $\beta^*\alpha^*(g)(a) = g(a)$, whence $\beta^*\alpha^*$ is the identity. Likewise $\alpha^*\beta^*$ is the identity, so $\mathrm{Aut}(M)$ and $\mathrm{Aut}(N)$ are isomorphic topological groups.

[1] After the lecture, G. Cherlin pointed out that Shelah explains how to do this in his book on Classification Theory. Consider M^3 quotiented by the definable equivalence relation:
$(c, a, b) \sim (c', a', b') \Leftrightarrow ((a = b) \wedge (a' = b') \wedge (c = c')) \vee ((a \neq b) \wedge (a' \neq b') \wedge (c = c'))$. Note that here, for each $c \in M$, the triples (c, a, b) fall into two equivalence classes.

Conversely, suppose $\gamma : \mathrm{Aut}(M) \to \mathrm{Aut}(N)$ is an isomorphism of topological groups. Let $a_1, \ldots, a_r$ be representatives of the $\mathrm{Aut}(N)$-orbits on N. For each $i \leq r$, there is $e_i \in M^{eq}$ such that $\gamma^{-1}(\mathrm{Aut}(M/e_i)) = \mathrm{Aut}(N/a_i)$ (by Theorem 2.7). If $a \in N$ there is a unique i and some $g \in \mathrm{Aut}(N)$ with $a = ga_i$. Define $\alpha(a) = \gamma^{-1}(g)(e_i)$. One checks that this is well-defined and the resulting $\alpha : N \to M^{eq}$ is an interpretation of N in M. Moreover, if $g \in \mathrm{Aut}(M)$ and $a \in N$ then $\alpha^*(g)(a) = \gamma(g)(a)$.

Similarly, we use γ to define an interpretation $\beta : M \to N^{eq}$ of M in N where $\beta^*(g) = \gamma^{-1}(g)$.

We claim that $\tilde{\alpha}\beta$ is definable in M^{eq}. To do this, it will suffice to show that if $g \in \mathrm{Aut}(M)$ fixes $a \in M$, then it fixes $\tilde{\alpha}\beta(a)$. Note that $\beta^*\alpha^*(g) = g$, so what we want follows from the computation of $\beta^*\alpha^*(g)(a)$ above. □

Example 2.13. Consider again the Example 2.5. We have an interpretation $\alpha : N \to M^{eq}$, but also the elements of M can be 'coded' as elements of N^{eq} in the following way. Note that the only way that four vertices in the graph $(N; R)$ can form a complete graph is if they share a common element of M. Let D be the definable subset of N^4 consisting of 4-tuples of distinct vertices forming a complete graph; let E be the definable relation on this such that $E(\bar{a}, \bar{b})$ holds iff there is a complete graph on the union of the points in $\bar{a}, \bar{b}$. This is a definable equivalence relation on D and the equivalence classes D/E correspond to the elements of M. One can then show that $\alpha^* : \mathrm{Aut}(M) \to \mathrm{Aut}(N)$ is a topological isomorphism.

3 The Hrushovski construction

3.1 An extension of Fraïssé's Theorem

We give a generalization of Fraïssé's Theorem 1.6. Further generalizations are possible (though the basic structure of the proof is always the same). For example a general category-theoretic version of the Fraïssé construction can be found in [9] and Section 2.6 of [21].

We shall work with a class $\mathcal{K}$ of finite L-structures and a distinguished class of substructures $A \sqsubseteq B$, pronounced 'A is a *nice* substructure of B' (the terminology is not standard). If $B \in \mathcal{K}$, then an embedding $f : A \to B$ is a $\sqsubseteq$-embedding if $f(A) \sqsubseteq B$. We shall assume that $\sqsubseteq$ satisfies:

(N1) If $B \in \mathcal{K}$ then $B \sqsubseteq B$ (so isomorphisms are $\sqsubseteq$-embeddings);

(N2) If $A \sqsubseteq B \sqsubseteq C$ (and $A, B, C \in \mathcal{K}$), then $A \sqsubseteq C$ (so if $f : A \to B$ and $g : B \to C$ are $\sqsubseteq$-embeddings, then $g \circ f : A \to C$ is a $\sqsubseteq$-embedding).

We say that $(\mathcal{K}, \sqsubseteq)$ is an *amalgamation class* if:

- $\mathcal{K}$ is closed under isomorphisms and has countably many isomorphism types (and countably many embeddings between any pair of elements);

- $\mathcal{K}$ is closed under $\sqsubseteq$-substructures;
- $\mathcal{K}$ has the JEP for $\sqsubseteq$-embeddings;
- $\mathcal{K}$ has AP for $\sqsubseteq$-embeddings: if A_0, A_1, A_2 are in $\mathcal{K}$ and $f_1 : A_0 \to A_1$ and $f_2 : A_0 \to A_2$ are $\sqsubseteq$-embeddings, there is $B \in \mathcal{K}$ and $\sqsubseteq$-embeddings $g_i : A_i \to B$ (for $i = 1, 2$) with $g_1 \circ f_1 = g_2 \circ f_2$.

Remarks 3.1. 1. If $\sqsubseteq$ is just 'substructure' then this is the same as what was previously defined as a (Fraïssé) amalgamation class.

2. The notion $A \sqsubseteq B$ is only defined when B is finite and it will be convenient to extend this to the situation where B is the union of a $\sqsubseteq$-chain of finite substructures. We can do this as follows.

   Suppose M is a countable L-structure and there are finite $M_i \subseteq M$ (with $i \in \mathbb{N}$) such that $M = \cup_{i \in \mathbb{N}} M_i$ and $M_1 \sqsubseteq M_2 \sqsubseteq M_3 \sqsubseteq \ldots$. Then for finite $A \subseteq M$ we define $A \sqsubseteq M$ to mean that $A \sqsubseteq M_i$ for some $i \in \mathbb{N}$. Note that *a priori* this depends on the choice of M_i, though the notation does not reflect this.

A condition on $(\mathcal{K}, \sqsubseteq)$ which guarantees that this *does not* depend on the choice of the M_i is:

(N3) Suppose $A \sqsubseteq B \in \mathcal{K}$ and $A \subseteq C \subseteq B$ with $C \in \mathcal{K}$. Then $A \sqsubseteq C$.

Indeed, suppose this holds and we also write M as the union of a $\sqsubseteq$-chain

$$M'_1 \sqsubseteq M'_2 \sqsubseteq M'_3 \sqsubseteq \ldots.$$

Suppose $A \sqsubseteq M_i$. There exist j, k such that

$$M_i \subseteq M'_j \subseteq M_k.$$

As $M_i \sqsubseteq M_k$ and $M'_j \in \mathcal{K}$, (N3) implies that $M_i \sqsubseteq M'_j$, so $A \sqsubseteq M'_j$.

The generalisation of the amalgamation construction is:

Theorem 3.2. *Suppose $(\mathcal{K}, \sqsubseteq)$ is an amalgamation class of finite L-structures and $\sqsubseteq$ satisfies (N1) and (N2). Then there is a countable L-structure M and finite substructures $M_i \in \mathcal{K}$ (for $i \in \mathbb{N}$) such that:*

1. *$M_1 \sqsubseteq M_2 \sqsubseteq M_3 \sqsubseteq \ldots$ and $M = \cup_{i \in \mathbb{N}} M_i$;*
2. *every $A \in \mathcal{K}$ is isomorphic to a $\sqsubseteq$-substructure of M;*
3. *(Extension Property) if $A \sqsubseteq M$ is finite and $f : A \to B \in \mathcal{K}$ is a $\sqsubseteq$-embedding then there is a $\sqsubseteq$-embedding $g : B \to M$ such that $g(f(a))$ for all $a \in A$.*

Moreover, M is determined up to isomorphism by these properties and if $A_1, A_2 \sqsubseteq M$ and $h : A_1 \to A_2$ is an isomorphism, then h extends to an automorphsim of M (which can be taken to preserve $\sqsubseteq$).

Note that in the above $\sqsubseteq$ is with respect to the chain of M_i. When we apply the result here, the $\sqsubseteq$ will satisfy ($N3$) so this dependence is irrelevant. We refer to the property in the 'Moreover' part as *$\sqsubseteq$-homogeneity* and say that M is the *generic structure* of the class $(\mathcal{K}, \sqsubseteq)$.

Proof of Theorem 3.2: This is very similar to the proof of Theorem 1.6 so we will only give an outline.

Existence of M: Build the M_i inductively ensuring that:

- if $C \in \mathcal{K}$ there is an i and a $\sqsubseteq$-embedding $f : C \to M_i$;
- if $A \sqsubseteq M_i$ and $A \sqsubseteq B \in \mathcal{K}$ then there is $j \geq i$ and a $\sqsubseteq$-embedding $g : B \to M_j$ with $g(a) = a$ for all $a \in A$.

To perform tasks of the first type, we use JEP; for the second type we can use AP as in the proof of 1.6. There are only countably many tasks to perform, so we can arrange that all are completed during the construction of the M_i.

Uniqueness and $\sqsubseteq$-homogeneity: Suppose $M'_1 \sqsubseteq M'_2 \sqsubseteq \ldots$ is a $\sqsubseteq$-chain whose union M' also satisfies (1-3). Write $\sqsubseteq'$ for $\sqsubseteq$ in M' with respect to the M'_i. As in the proof of 1.6, one uses the Extension Property to show that

$$\mathcal{S} = \{f : A \to A' : f \text{ an isomorphism and } A \sqsubseteq M,\ A' \sqsubseteq' M' \text{ finite}\}$$

is a back-and-forth system (which is non-empty because of JEP).

It follows that if $f : A \to A'$ is in $\mathcal{S}$ then there is an isomorphism $h : M \to M'$ which extends f. Moreover, the back-and-forth construction of h will ensure that $h(M_i) \sqsubseteq' M'$ and $h^{-1}(M'_i) \sqsubseteq M$ for all i, so $h(B) \sqsubseteq' M' \Leftrightarrow B \sqsubseteq M$ (for finite $B \subseteq M$). □

As with Theorem 1.6, there is a converse statement. We omit the proof.

Theorem 3.3. *Suppose M is a countable L-structure and $(\mathcal{K}, \sqsubseteq)$ (satisfying (N1), (N2)) is such that $M = \cup_{i \in \mathbb{N}} M_i$ for $M_i \in \mathcal{K}$ with $M_i \sqsubseteq M_{i+1}$. Suppose also that $\mathcal{K}$ is the class of isomorphism types of $\sqsubseteq$-substructures of M and that M is $\sqsubseteq$-homogeneous (with respect to the $\sqsubseteq$-chain). Then Then $(\mathcal{K}, \sqsubseteq)$ is an amalgamation class.* □

Remarks 3.4. Suppose $\mathcal{K}$ in Theorem 3.2 has only finitely many isomorphism types of structure of each finite size. Suppose also that there is a function $F : \mathbb{N} \to \mathbb{N}$ such that if $B \in \mathcal{K}$ and $A \subseteq B$ with $|A| \leq n$, then there is $C \sqsubseteq B$ with $A \subseteq C$ and $|C| \leq F(n)$. Then the generic structure M is ω-categorical.

To see this we note that $\mathrm{Aut}(M)$ has finitely many orbits on M^n. Indeed, by $\sqsubseteq$-homogeneity there are finitely many orbits on $\{\bar{c} \in M^{F(n)} : \bar{c} \sqsubseteq M\}$ and any $\bar{a} \in M^n$ can be extended to an element of this set.

3.2 Hrushovski's predimension construction

Notation 3.5. Fix an integer $n \geq 2$ and a positive real number α. Let L be a language with a single n-ary relation symbol R. We work with the class $\bar{\mathcal{C}}$ of L-structures A where R is symmetric and irreflexive. Thus, the set R^A of instances of R in A can be thought of as a set of n-subsets from A, so $R^A \subseteq [A]^n$. If A is finite, define the *predimension* of A to be:

$$\delta(A) = \alpha|A| - |R^A|.$$

Remarks 3.6.

1. This can be done more generally. The condition of R being symmetric is not really necessary (but for example, if $n = 2$, then I prefer to work with graphs rather than directed graphs). We can also have a family of atomic relations $(R_i : i \in I)$ in L and work with a predimension $\delta(A) = \alpha|A| - \sum_{i\in I} \beta_i |R_i^A|$ where $\beta_i \geq 0$.
2. If $\alpha = m/\ell \in \mathbb{Q}$ we will rescale δ as $\delta(A) = m|A| - \ell|R^A|$.

Lemma 3.7. *(Submodularity) Suppose $A \in \bar{\mathcal{C}}$ and B, C are finite subsets of A. Then*

$$\delta(B \cup C) \leq \delta(B) + \delta(C) - \delta(B \cap C).$$

Moreover, there is equality here iff $R^{B\cup C} = R^B \cup R^C$ (that is, B are freely amalgamated over $B \cap C$ in A).

Proof. Note that

$$\begin{aligned}\delta(B) + \delta(C) - \delta(B \cap C) - \delta(B \cup C) \\ &= -(|R^B| + |R^C| - |R^{B\cap C}| - |R^{B\cup C}|) \\ &= |R^{B\cup C}| - (|R^B| + |R^C| - |R^B \cap R^C|) \\ &= |R^{B\cup C}| - |R^B \cup R^C| \geq 0\end{aligned}$$

with equality iff $R^{B\cup C} = R^B \cup R^C$. □

Definition 3.8. Suppose $B \in \bar{\mathcal{C}}$ and $A \subseteq B$ is finite.

1. Write $A \leq_s B$ if $\delta(A) \leq B'$ for all finite B' with $A \subseteq B' \subseteq B$ and say that A is *self-sufficient* in B.
2. Write $A \leq_d B$ if $\delta(A) < \delta(B')$ for all finite B' with $A \subset B' \subseteq B$ and say that A is *d-closed* in B.

Remarks 3.9.

1. Take $n = 2$ and $\alpha = 2$. So $\delta(A)$ is 'twice number of vertices minus number of edges'. If B consists of 3 vertices a, b, c with edges ab, bc and $A = \{a, c\}$, then $A \leq_s B$ but $A \not\leq_d B$.
2. In general, if $A \leq_d B$ then $A \leq_s B$.

3. If α is irrational and B finite, then $A \leq_s B$ implies $A \leq_d B$. Otherwise there is $A \subset B' \subseteq B$ with $\delta(A) = \delta(B')$. Then $\alpha = (|R^{B'}| - |R^A|)/(|B'| - |A|)$, which is rational.

Lemma 3.10. *Let $B \in \bar{\mathcal{C}}$ and let $\leq$ denote either $\leq_s$ or $\leq_d$.*

1. *If $A \leq B$ is finite and $X \subseteq B$, then $A \cap X \leq X$.*
2. *If A, C are finite and $A \leq C \leq B$, then $A \leq B$.*
3. *If A_1, A_2 and finite and $A_1, A_2 \leq B$, then $A_1 \cap A_2 \leq B$.*

Proof. (1) Let $Y \subseteq X$ be finite with $A \cap X \subset Y$. Note that $Y \cap A = X \cap A$. Then by submodularity:

$$\delta(A \cup Y) \leq \delta(A) + \delta(Y) - \delta(Y \cap A) \text{ so } \delta(A \cup Y) \leq \delta(A) + \delta(Y) - \delta(X \cap A).$$

Therefore

$$\delta(Y) - \delta(A \cap X) \geq \delta(A \cup Y) - \delta(A).$$

If $A \leq_s B$, this is ≥ 0. If $A \leq_d B$, it is > 0.

(2) We give the proof for $\leq_s$; the proof for $\leq_d$ is similar. Let $A \subset X \subseteq_{fin} B$. By (1), $C \cap X \leq_s X$ so

$$\delta(A) \leq \delta(X \cap C) \leq \delta(X)$$

(the first of these coming from $A \leq_s C$).

(3) By (1) $A_1 \cap A_2 \leq A_1$, so the result follows from (2). □

Remarks 3.11. If B is finite, then (3) shows that if $A \subseteq B$ and $S = \{A_1 : A \subseteq A_1 \leq B\}$, then $\bigcap S \leq B$. So there is a smallest $\leq_s$ (or $\leq_d$) subset of B which contains A: denote it by $\mathrm{cl}^s_B(A)$ (or $\mathrm{cl}^d_B(A)$) respectively. It is easy to see that cl^s_B and cl^d_B are closure operations on B.

Lemma 3.12. *For finite $A \subseteq B \in \bar{\mathcal{C}}$ we have $\delta(A) \geq \delta(\mathrm{cl}^d_B(A))$.*

Proof. Amongst all the subsets X of B containing A, consider the ones for which $\delta(X)$ is as small as possible. Amongst these, choose one, C, with as many elements as possible. Clearly $\delta(C) \geq \delta(A)$ and if $C \subset D \subseteq B$, then $\delta(C) < \delta(D)$. So $C \leq_d B$ and therefore $A \subseteq \mathrm{cl}^d_B(A) \leq_d C \leq_d B$. By choice of C we have $\delta(C) \leq \delta(\mathrm{cl}^d_B(A))$, therefore $C = \mathrm{cl}^d_B(A)$. The result follows. □

From Lemma 3.10, the notions of distinguished substructure $\leq_s$ and $\leq_d$ satisfy (N1), (N2) and (N3) (for the finite structures in $\bar{\mathcal{C}}$). However, we do not have the JEP. For example, suppose $B_1, B_2 \in \bar{\mathcal{C}}$ are finite and $\delta(B_2) < 0$. Let C be the free amalgam (disjoint union) of B_1 and B_2. Then $\delta(B_1) + \delta(B_2) < \delta(B_1)$ so $B_1 \not\leq_s C$. So it makes sense to exclude structures of negative predimension.

Definition 3.13. Let

$$\bar{\mathcal{C}}_0 = \{B \in \bar{\mathcal{C}} : \emptyset \leq_s B\}$$

and

$$\bar{\mathcal{C}}_{>0} = \{B \in \bar{\mathcal{C}} : \emptyset \leq_d B\}.$$

So

$$B \in \bar{\mathcal{C}}_0 \text{ iff for all non-empty, finite } A \subseteq B \text{ we have } \delta(A) \geq 0,$$

and

$$B \in \bar{\mathcal{C}}_{>0} \text{ iff for all non-empty, finite } A \subseteq B \text{ we have } \delta(A) > 0.$$

Let $\mathcal{C}_0$ and $\mathcal{C}_{>0}$ denote the finite members of these classes.

Remarks 3.14. If $A \in \mathcal{C}_0$ then $|R^A| \leq \alpha|A|$: the structures in $\mathcal{C}_0$ are *sparse*.

For example, if $n = 2$, $\alpha = 1$ and $A \in \mathcal{C}_0$, then each connected component of A has at most one cycle, so this case is not very interesting. The case $\alpha = 2$ is more interesting.

Lemma 3.15. *The classes $(\mathcal{C}_{>0}, \leq_d)$ and $(\mathcal{C}_0, \leq_s)$ are free amalgamation classes and (N1), (N2), (N3) hold.*

Proof. We verify this for $(\mathcal{C}_{>0}, \leq_d)$: the proof for the other class is almost identical.

It remains to consider the amalgamation property (the JEP is a special case of this). We show the following stronger form. Let $A \subseteq B_1, B_2 \in \mathcal{C}_{>0}$ and $A \leq_d B_1$. Let E be the free amalgam of B_1 and B_2 over A. We claim that $B_2 \leq_d E$. Note that once we have the claim, we have $\emptyset \leq_d B_2 \leq_d E$, so $\emptyset \leq_d E$ and therefore $E \in \mathcal{C}_{>0}$.

To establish the claim, let $B_2 \subset X \subseteq E$. So $X = B_2 \cup Y$ where $Y = X \cap B_1 \supset A$. Moreover, X is the free almalgam of B_2 and Y over A, so

$$\delta(X) = \delta(B_2 \cup Y) = \delta(B_2) + \delta(Y) - \delta(A).$$

So

$$\delta(X) - \delta(B_2) = \delta(Y) - \delta(A) > 0$$

as $A \leq_d B_1$. □

Remarks 3.16. Using Theorem 3.2, we can build a countable generic structure $M_{>0}$ for $(\mathcal{C}_{>0}, \leq_d)$ (and M_0 for $(\mathcal{C}_0, \leq_s)$). However, neither of these will be ω-categorical: if $A \subseteq M_{>0}$ is finite there is no uniform bound on the size of $\mathrm{cl}^d(A)$ in terms of $|A|$ (and there is a similar problem in M_0).

Remarks 3.17.

1. The theory of M_0 is stable (and ω-stable if α is rational).
2. If $n = 2$ and α is irrational then $Th(M_0)$ is the limit theory as $k \to \infty$ of random finite graphs on k vertices where edges are chosen with probability $k^{-1/\alpha}$ ([5, 26]).
3. In general if α is rational then $Th(M_{>0})$ is bad. For example if $n = 2$ and $\alpha = 2$ then $Th(M_{>0})$ has the independence and strict order properties; it encodes the theory of finite graphs, so is undecidable.

3.3 Construction of ω-categorical examples

Following [14], we will consider subclasses of $(\mathcal{C}_{>0}, \leq_d)$ in which d-closure is uniformly bounded. More precisely we have the following definition.

Definition 3.18. Let $F : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ be a continuous, increasing function with $F(x) \to \infty$ as $x \to \infty$, and $F(0) = 0$. Let

$$\mathcal{C}_F = \{B \in \mathcal{C}_{>0} : \delta(A) \geq F(|A|) \text{ for all } A \subseteq B\}.$$

Theorem 3.19. *1. If $B \in \mathcal{C}_F$ and $A \subseteq B$ then*

$$|\mathrm{cl}^d_B(A)| \leq F^{-1}(\alpha|A|).$$

2. If $(\mathcal{C}_F, \leq_d)$ is an amalgamation class, then the generic structure M_F is ω-categorical.

3. If $(\mathcal{C}_F, \leq_d)$ is a free amalgamation class and $A \subseteq M_F$ is finite, then

$$\mathrm{acl}_{M_F}(A) = \mathrm{cl}^d_{M_F}(A).$$

Proof. (1) By Lemma 3.12 we have $\delta(\mathrm{cl}^d_B(A)) \leq \delta(A) \leq \alpha|A|$. Thus (by definition of $\mathcal{C}_F$) we have $|\mathrm{cl}^d_B(A)| \leq F^{-1}(\alpha|A|)$.

(2) This follows from Remarks 3.4.

(3) As $\mathrm{cl}^d_{M_F}(A)$ is finite and invariant under $\mathrm{Aut}(M_F/A)$, it is contained in $\mathrm{acl}_{M_F}(A)$. On the other hand, if $B \leq_d M_F$ is finite and $B \leq_d C \leq_d M_F$ (C finite), then for $c \in C \setminus B$ the $\mathrm{Aut}(M_F/B)$-orbit containing c is infinite. The proof of this is as in the proof of Lemma 1.18. Using free amalgamation and the Extension Property for M_F, we see that M_F contains a copy (over B) of the free amalgam of k distinct copies of C. The copies of c inside these are all in the same $\mathrm{Aut}(M_F/B)$-orbit as c. As k is arbitrary here, we obtain the result. □

Example 3.20. Let $m, \ell \in \mathbb{N}$ and $\alpha = m/\ell$. Work with the integer-valued predimension $\delta(A) = m|A| - \ell|R^A|$. Let F as in Definition 3.18 be such that:

- F is piecewise smooth;
- the right derivative F' is non-increasing;
- $F'(x) \leq 1/x$ for all $x > 0$.

The we claim that $(\mathcal{C}_F, \leq_d)$ is a free amalgamation class.

Indeed, suppose $A \leq_d B_1, B_2 \in \mathcal{C}_F$ and let E be the free amalgam of B_1 and B_2 over A. We need to show that $E \in \mathcal{C}_F$. Clearly we may assume $A \neq B_i$.

Suppose $X \subseteq E$. We need to show that $\delta(X) \geq F(|X|)$. Now, X is the free amalgam over $A \cap X$ of $B_1 \cap X$ and $B_2 \cap X$ and $A \cap X \leq_d B_i \cap X$ (by Lemma 3.10(1)). So we can assume $X = E$ and check that $\delta(E) \geq F(|E|)$.

Note that $\delta(E) = \delta(B_1) + \delta(B_2) - \delta(A)$ and $|E| = |B_1| + |B_2| - |A|$.

The effect of the conditions on F is that for $x, y \geq 0$

$$F(x+y) \leq F(x) + yF'(x) \leq F(x) + y/x.$$

We can assume that

$$\frac{\delta(B_2) - \delta(A)}{|B_2| - |A|} \geq \frac{\delta(B_1) - \delta(A)}{|B_1| - |A|}$$

and note that the latter is at least $1/|B_1|$ (as δ is integer-valued and $A \leq_d B_1$).

Then

$$\delta(E) = \delta(B_1) + (|B_2| - |A|)\frac{\delta(B_2) - \delta(A)}{|B_2| - |A|}$$
$$\geq F(|B_1|) + (|B_2| - |A|)/|B_1| \geq F(|E|)$$

(taking $x = |B_1|$ and $y = |B_2| - |A|$).

This concludes the proof of the claim.

Example 3.21. We use this to produce an example of a connected ω-categorical graph whose automorphism group is transitive on vertices and edges, and whose smallest cycle is a 5-gon.

Let $n = 2$, $\alpha = 2$. So we are working with graphs and the predimension:

$$\delta(A) = 2|A| - |R^A|.$$

Take

$$F(1) = 2; F(2) = 3; F(5) = 5; F(k) = \log(k) + 5 - \log(5) \text{ for } k \geq 5.$$

Then one can check that:

- The smallest cycle in $\mathcal{C}_F$ is a 5-gon;
- If $a \in A \in \mathcal{C}_F$ then $a \leq_d A$
- If $ab \subseteq B \in \mathcal{C}_F$ is an edge then $ab \leq_d B$
- $(\mathcal{C}_F, \leq_d)$ is an amalgamation class (the proof of AP in the previous example applies if at least one of B_1, B_2 has size ≥ 5; the other cases can be checked individually).
- The Fraïssé limit M_F is connected. Given non-adjacent $a, b \in M_F$ consider $A = \mathrm{cl}^d(ab)$. As $\delta(A) \leq \delta(ab) = 4$ we have $|A| \leq 3$. So either A is a path of length 2 (with endpoints a, b) or $A = ab$, so $ab \leq_d M_F$. In the latter case, consider a path B of length 3 with end points a, b. Then $ab \leq_d B$ so there is a $\leq_d$ copy of B in M_F over ab. In particular, ab are at distance 3 in M_F.

It follows that the smallest cycle in the Fraïssé limit M_F is a 5-cycle and $\mathrm{Aut}(M_F)$ is transitive on vertices and edges. In fact, the argument shows that M_F is distance transitive of diameter 3.

Example 3.22. Take $\alpha > 1$ irrational. Let $A \leq_d B$ (with $A \neq B$) and note that

$$0 < \delta(B) - \delta(A) = \alpha|B \setminus A| - |R^B \setminus R^A| = (|B| - |A|)(\alpha - |R^B \setminus R^A|/|B \setminus A|).$$

So

$$0 < (\delta(B) - \delta(A))/(|B| - |A|) = \alpha - q$$

for some $q \in \mathbb{Q}$, $q > 0$.

For any $N \in \mathbb{N}$ let

$$\eta(\alpha, N) = \min(\alpha - \frac{a}{b} : b \leq N, \frac{a}{b} \leq \alpha).$$

Note that $\eta(\alpha, N) \geq \eta(\alpha, N+1)$ and in the above:

$$(\delta(B) - \delta(A))/(|B| - |A|) \geq \eta(\alpha, |B|).$$

Claim: α can be chosen so that $\sum_{N=1}^{\infty} \eta(\alpha, N)$ diverges.

Suppose we have the claim. For such an α let $F_\alpha : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ be the piecewise linear hull with $F_\alpha(N) = \sum_{k=1}^{N} \eta(\alpha, k)$.

Then the proof in Example 3.20 shows that $(\mathcal{C}_{F_\alpha}, \leq_d)$ is an amalgamation class and the generic structures M_{F_α} is ω-categorical.

One way to establish the claim is to use the Baire Category Theorem. For any α let

$$i(\alpha) = \sum_{N=1}^{\infty} \eta(\alpha, N)$$

allowing $i(\alpha) = \infty$.

One shows that for each $M \in \mathbb{N}$, the set

$$\{\alpha > 1 : i(\alpha) > M\}$$

is open and dense in the interval $(1, \infty)$.

3.4 Some history

The Hrushovski construction first appeared in the unpublished notes [14, 15]. In [14], Hrushovski produces a strictly stable ω-categorical structure, giving a counterexample to Lachlan's conjecture (Example 3.22 here). The paper also contains the construction of an ω-categorical pseudoplane using an integer-valued predimension, a control function and d-closed embeddings (Example 3.21). The notes [15], which were extended into the paper [17], construct a strongly minimal set which is a counterexample to Zilber's conjecture. An intermediate stage in the construction gives a new ω-stable structure of infinite Morley rank (the strutures M_0 in Remarks 3.16 here). A survey of this material which gives an axiomatic treatment of the construction under the assumption of finiteness of self-sufficient closure is given by Wagner in [28]. An axiomatic treatment without assuming finiteness of self-sufficient closure is given by Baldwin and Shi [1].

In unpublished notes [18] Hrushovski showed that a further condition on the function F in Example 3.20 produces supersimple ω-categorical structures M_F. An axiomatic treatment is given in [12] and a direct approach is given in Section 6.2.1 of Wagner's book [29]. The latter gives a different proof of simplicity (using the Kim - Pillay characterization in terms of existence of a suitable notion of independence).

Further model-theoretic applications of the construction given by Hrushovski are the construction of strongly minimal sets [17] and the fusion construction [16]. The former requires an additional amalgamation result ('algebraic amalgamation' in [17]; now usually called the 'collapse'). The latter produces, for example, a strongly minimal structure having as reducts two algebraically closed fields of different characteristics. A very readable exposition of this is given in [4].

The construction has been used to produce algebraic structures of finite Morley rank which differ from classical examples (bad fields [3]; non-algebraic nilpotent groups [2]). Moreover, there are many surprising and subtle conections between the construction and other areas of mathematics. For example: random graphs [26, 5], transcendence theory and complex exponentiation [30], and the papers [21], [8].

4 A short Appendix on Model Theory

4.1 First-order languages and structures

In a first-order language one has an alphabet of symbols and certain finite sequences of these symbol (the formulas of the language) are the objects of interest. The symbols are connectives $\wedge$ (*and*), $\vee$ (*or*), $\neg$ (*not*); quantifiers $\forall$ and $\exists$; punctuation (parentheses and commas); variables; and constant, relation and function symbols, with each of the last two coming equipped with a finite 'arity' specifying how many arguments it has. The number of these constant, relation and function symbols (together with their arities) is referred to as the *signature* of the language.

The *terms* of the language are built inductively. Any variable or constant symbol is a term and if f is an n-ary function symbol and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is also a term (all terms are built in this way).

Now we can build the *formulas* of the language. Again, this is done inductively. If R is an n-ary relation symbol in the language and $t_1, \ldots, t_n$ are terms then $R(t_1, \ldots, t_n)$ is a formula (an *atomic* formula). If ϕ, ψ are formulas and x a variable, then $(\phi) \wedge (\psi)$, $(\phi) \vee (\psi)$, $\neg(\phi)$, $\forall x(\phi)$, $\exists x(\phi)$ are formulas (of higher 'complexity'). A formula not involving any quantifiers is called *quantifier free* or *open*. There is a natural notion of a *free variable* in a formula, and when we write a formula as $\phi(x_1, \ldots, x_m)$ we mean that its free variables are amongst the variables $x_1, \ldots, x_m$. A formula with no free variables is called a *sentence*. For more details the reader could consult ([13], Section 2.1).

If L is a first-order language then an *L-structure* consists of a set M equipped with a constant (that is, a distinguished element of M), n-ary relation (that is, a subset of M^n), and n-ary function $M^n \to M$ for each constant symbol and n-ary relation and function symbol in L. If $\phi(x_1, \ldots, x_m)$ is an L-formula and $a_1, \ldots, a_m \in M$ then one can 'read' $\phi(a_1, \ldots, a_m)$ as a statement about the behaviour of $a_1, \ldots, a_m$ and these constants, relations and functions (interpreting each constant, relation or function symbol as the corresponding constant, relation or function of M), which is either true or false. If it is true, then we write

$$M \models \phi(a_1, \ldots, a_m).$$

All of this can of course be made completely precise (defined inductively on the complexity of ϕ): see ([13], Section 2.1) again. We shall always have $=$ as a binary relation symbol in L and interpret it as true equality in any L-structure.

If Φ is a set of L-sentences and M an L-structure we say that M is a *model* of Φ (and write $M \models \Phi$) if every sentence in Φ is true in M. If there is a model of Φ we say that Φ is *consistent*. The set of L-sentences true in M is called the *theory* of M. Two L-structures M_1 and M_2 are *elementarily equivalent* if they have the same theory. This is written as $M_1 \equiv M_2$. Thus in this case the structures M_1 and M_2 cannot be distinguished using the language L. The following basic result of model theory shows that one should not expect first-order languages to be able to completely describe infinite structures.

Theorem 4.1. (Löwenheim-Skolem) *Let L be a first-order language with signature of cardinality λ. Let μ, ν be cardinals with $\mu, \nu \geq \max(\lambda, \aleph_0)$, and suppose M_1 is an L-structure with cardinality μ. Then there exists an L-structure M_2 elementarily equivalent to M_1 and of cardinality ν.*

The 'upward' part of this result (where $\nu \geq \mu$) follows easily from the fundamental theorem of model theory:

Theorem 4.2. (The Compactness Theorem) *Let L be a first-order language and Φ a set of L-sentences. If every finite subset of Φ is consistent, then Φ is consistent.*

The original version of this is due to Gödel (1931). Proofs (using a method due to Henkin (1949)) can be found in ([13], Theorem 6.1.1). Algebraists may prefer the proof using ultraproducts and the theorem of Łos ([13], Theorem 9.5.1).

If M, N are L-structures with $M \subseteq N$ and the distinguished relations, functions (and constants) of N extend those of M, then we say that M is a *substructure* of N. If also for every L-formula $\phi(x_1, \ldots, x_m)$ and $a_1, \ldots, a_m \in M$ we have

$$M \models \phi(a_1, \ldots, a_m) \Leftrightarrow N \models \phi(a_1, \ldots, a_m)$$

then we say that M is an *elementary* substructures of N (and that N is an elementary extension of M) and write $M \preceq N$. A stronger version of the Löwenheim-Skolem Theorem (4.1) is true: the smaller of M_1, M_2 may be taken to be an elementary substructure of the larger. Proofs can be found in ([13], Corollaries 3.1.5 and 6.1.4).

4.2 Definable sets; types

Suppose L is a first-order language and M an L-structure. Let $n \in \mathbb{N}$. A subset A of M^n is called (parameter) *definable* if there exist $b_1, \ldots, b_m \in M$ and an L-formula

$\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ with

$$A = \{\bar{a} \in M^n : M \models \phi(\bar{a}, \bar{b})\}.$$

If the parameters $\bar{b}$ can be taken from the subset $X \subseteq M$ then A is said to be X-definable. The union of the finite X-definable subsets of M is called the *algebraic closure* of X, denoted by $\mathrm{acl}(X)$, and the union of the X-definable singleton subsets of M is the *definable closure* of X, denoted by $\mathrm{dcl}(X)$. It is not hard to check that both of these are indeed closure operations on M.

So the definable subsets of M^n are the ones which can be described using L-formulas (and parameters). Conversely one could take a particular n-tuple $\bar{a} \in M^n$ and a set of parameters $A \subseteq M$ and ask what the language L can say about $\bar{a}$ (in terms of A and M). This gives the notion of the *type* of $\bar{a}$ over A, which by definition is

$$\mathrm{tp}^M(\bar{a}/A) = \{\phi(x_1, \ldots, x_n, b_1, \ldots, b_m) : b_1, \ldots, b_m \in A,\ M \models \phi(\bar{a}, \bar{b})\}$$

(the superscript M is dropped if this is clear from the context). It is sometimes useful to consider the type of $\bar{a}$ (over A) using only certain L-formulas. For example, for the *quantifier free type* of $\bar{a}$ over A one takes only quantifier free ϕ in the above definition. It is also possible to define the type of an infinite sequence of elements of M. The reader can consult ([13], Section 6.3) for further details here.

More generally, a (complete) n-type over A is a set of L-formulas with parameters from A equal to $\mathrm{tp}_N(\bar{a}/A)$ for some elementary extension N of M and some $\bar{a} \in M^n$. There is no reason to suppose, for arbitrary M and A, that this type should be *realised* in M, that is, there exists $\bar{a}' \in M^n$ with $\mathrm{tp}^M(\bar{a}'/A) = \mathrm{tp}^N(\bar{a}/A)$. For example, this would clearly be impossible if $A = M$ and $\bar{a} \notin M^n$. However, it can happen that for some infinite cardinal κ if $|A| < \kappa$ then every complete n-type over A is realised in M: in this case M is called *κ-saturated,* and if $\kappa = |M|$ then M is *saturated.* The reader should consult ([13], Chapter 10) for more on this.

References

[1] John T. Baldwin and Niandong Shi, Stable generic structures, Annals of Pure and Applied Logic 79 (1996), 1–35.

[2] Andreas Baudisch, A new uncountably-categorical group, Transactions of the AMS 348 (1996), 3889–3940.

[3] Andreas Baudisch, Martin Hils, Amador Martin-Pizarro, Frank O. Wagner, Die böse Farbe , J. Inst. Math. Jussieu 8(3) (2007), 415–443.

[4] Andreas Baudisch, Amador Martin-Pizarro, Martin Ziegler, Hrushovski's fusion, in: Frieder Haug, Benedikt Löwe, Torsten Schatz (eds.), Algebra, Logic, Set Theory, Festschrift für Ulrich Felgner zum 65. Geburtstag; College Publications, Studies in Logic, Vol. 4, London 2007, 15–31.

[5] John T. Baldwin and Saharon Shelah, 'Randomness and semigenericity', Transactions of the American Mathematical Society 349 (1997), 1359–1376.

[6] Peter J. Cameron, Oligomorphic Permutation Groups, London Mathematical Society Lecture Notes, Vol. 152 Cambridge University Press, 1990.

[7] Gregory Cherlin, The classification of countable homogeneous directed graphs and countable n-tournaments, Memoirs of the American Math. Soc. 131 (1998), 621.

[8] Olivier Chapuis, Ehud Hrushovski, Pascal Koiran and Bruno Poizat, 'La limite des théories de courbes génériques', J. Symbolic Logic 57 (2002), 24–34.

[9] Manfred Droste and Rüdiger Göbel, A categorical theorem on universal objects and its application in abelian group theory and computer science, *Contemporary Mathematics* **131** (1992), 49–74.

[10] David M. Evans, Examples of $\aleph_0$-categorical structures, In [19], pp. 33–72.

[11] David M. Evans, Model Theory of Groups and Automorphism Groups. Proceedings of a Summer School in Blaubeuren, Germany, August 1995. London Mathematical Society Lecture Notes Series, 244, Cambridge University Press, 1997.

[12] David M. Evans, '$\aleph_0$-categorical structures with a predimension', Annals of Pure and Applied Logic 116 (2002), 157–186.

[13] Wilfrid Hodges, Model theory, Volume 42 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1993.

[14] Ehud Hrushovski, 'A stable $\aleph_0$-categorical pseudoplane', Unpublished notes, 1988.

[15] Ehud Hrushovski, 'A new strongly minimal set', Unpublished notes, 1988.

[16] Ehud Hrushovski, 'Strongly minimal expansions of algebraically closed fields', Israel J. Math. 79 (1992), 129–151.

[17] Ehud Hrushovski, A new strongly minimal set, Annals of Pure and Applied Logic 62 (1993) 147–166.

[18] Ehud Hrushovski, 'Simplicity and the Lascar group', Unpublished notes, December 1997.

[19] R. Kaye and D. Macpherson (eds), Automorphism Groups of First-order Structures, Oxford University Press, Oxford, 1994.

[20] Alexander S. Kechris, Dynamics of non-archimedean Polish groups, ...

[21] Jonathan Kirby, The theory of the exponential differential equations of semiabelian varieties, *Selecta Mathematica New Ser.* **15** (2009), 445–486.

[22] D. W. Kueker and M. C. Laskowski, 'On generic structures', Notre Dame J. Formal Logic 33 (1992), 175–183.

[23] Dugald Macpherson, Homogeneous structures, MALOA Workshop, Fischbachau, September 2010 (lecture notes on MALOA website).

[24] H. Dugald Macpherson, A survey of homogeneous structures, Discrete Mathematics 311 (2011), 1599–1634.

[25] David Marker, Model Theory: An Introduction, Springer GTM 217, Springer, New York, 2002.

[26] Saharon Shelah and Joel Spencer, 'Zero-one laws for sparse random graphs', Journal of the Amer. Math. Soc. 1 (1988), 97–115.

[27] Katrin Tent and Martin Ziegler, A Course in Model Theory, ASL Lecture Notes in Logic, Cambridge University Press, Cambridge 2012.

[28] Frank O. Wagner. Relational structures and dimensions. In [19], pp. 153–180.

[29] Frank O. Wagner, Simple Theories, Kluwer, Dordrecht, 2000.

[30] Boris Zilber, 'Pseudo-exponentiation on algebraically closed fields of characteristic zero', Annals Pure and Applied Logic 132 (2005), 67–95.

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