Spaces with Ricci curvature bounded from below

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1) On the definition of spaces with Ricci curvature bounded from below

2) Analytic properties of $RCD^*(K, N)$ spaces

3) Geometric properties of $RCD^*(K, N)$ spaces

4) More on the differential structure of metric measure spaces
Content

- First order differential structure of metric measure spaces
  - $L^2$-normed $L^\infty$-modules
  - The construction
  - Behavior under transformations

- Second order differential structure of RCD spaces
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- First order differential structure of metric measure spaces
  - $L^2$-normed $L^\infty$-modules
  - The construction
  - Behavior under transformations

- Second order differential structure of RCD spaces
The setting

$(\mathcal{X}, d, m)$ is such that:

- $(\mathcal{X}, d)$ is complete and separable
- $m$ is a non-negative Radon measure on it
$L^2$-normed $L^\infty(m)$-modules

An $L^2$-normed $L^\infty(m)$-module is given by a Banach space $(M, \| \cdot \|_M)$ equipped with:

1. a multiplication with $L^\infty(m)$ functions, i.e. a bilinear map $L^\infty(m) \times M \to M$ satisfying $f(gv) = (fg)v$, for every $f, g \in L^\infty(m)$ and $v \in M$.
2. a pointwise $L^2$-norm, i.e. a map $|\cdot|: M \to L^2(m)$ satisfying $|v| \geq 0$, $m$-a.e., $|fv| = |f||v|$, $m$-a.e., and $\|v\|_M = \sqrt{\int |v|^2 \, dm}$. 
An $L^2$-normed $L^\infty (\mathfrak{m})$-module is given by a Banach space $(M, \| \cdot \|_M)$ equipped with:

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f(gv) = (fg)v,
\]

\[
1v = v,
\]

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**$L^2$-normed $L^\infty(\mathfrak{m})$-modules**

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  $$
  
  $$
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  $$
  
  $$
  \|v\|_M = \sqrt{\int |v|^2 \, dm}
  $$
Why such a notion

**Basic example:**
The space of $L^2$ vector fields on a Riemannian manifold.

More generally, the space of $L^2$ sections of a normed vector bundle.
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More generally, the space of $L^2$ sections of a normed vector bundle.

**The idea:**
In the smooth setting, one can fully describe a vector bundle by looking either to its fibers, or to its sections.

In the non-smooth one, we take the latter viewpoint and thus declare $L^2$-normed $L^\infty(m)$-modules to ‘be’ vector bundles on our metric measure space.
Basic features of modules: locality

For $v, w \in M$ and a Borel set $E \subset \mathcal{X}$ we say that

$$v = w, \quad m - a.e. \text{ on } E$$

provided

$$\chi_E(v - w) = 0.$$ 

or equivalently

$$|v - w| = 0, \quad m - a.e. \text{ on } E.$$
Basic features of modules: duality

The dual $M^*$ of $M$ is the space of linear continuous maps $L : M \to L^1(\mathfrak{m})$ which are local, i.e. such that

$$L(fv) = f L(v), \quad \forall v \in M, \ f \in L^\infty(\mathfrak{m}).$$

$M^*$ is also an $L^2$-normed $L^\infty$-module, the pointwise norm being given by

$$|L|_* := \text{ess-sup} \ L(v) \quad v : |v| \leq 1 \ \mathfrak{m} - \text{a.e.}$$
Example: the dual of $L^2(m)$

The dual of $L^2(m)$ as Hilbert space is $L^2(m)$, i.e. for $L : L^2(m) \to \mathbb{R}$ linear and continuous there is a unique $g \in L^2(m)$ such that

$$L(f) = \int fg \, dm \quad \forall f \in L^2(m)$$

and $\|L\|_{L^2(m)'} = \|g\|_{L^2(m)}$. And viceversa.
Example: the dual of $L^2(\mathfrak{m})$

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and $\|L\|_{L^2(\mathfrak{m})'} = \|g\|_{L^2(\mathfrak{m})}$. And viceversa.

The dual of $L^2(\mathfrak{m})$ as module is $L^2(\mathfrak{m})$, i.e. for $T : L^2(\mathfrak{m}) \to L^1(\mathfrak{m})$ linear, continuous and local, there is a unique $g \in L^2(\mathfrak{m})$ such that

$$T(f) = fg \quad \mathfrak{m} - \text{a.e.} \quad \forall f \in L^2(\mathfrak{m})$$

and $|T|_* = |g| \quad \mathfrak{m}\text{-a.e.}$. And viceversa.
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Calculus rules for $|Df|$ 

Lower semicontinuity:

$$\left\{ \begin{array}{l}
(f_n) \subset S^2(\mathcal{X}) \\
 f_n \to f \quad \text{m - a.e.} \\
|Df_n| \to G \text{ in } L^2(\text{m}) \\
\end{array} \right\} \Rightarrow \left\{ \begin{array}{l}
 f \in S^2(\mathcal{X}) \\
|Df| \leq G \\
\end{array} \right.$$ 

Subadditivity: $|D(\alpha f + \beta g)| \leq |\alpha||Df| + |\beta||Dg|$ \quad m - a.e.

Locality: $|Df| = |Dg|$ \quad m - a.e. on $\{f = g\}$

Chain rule: $|D(\varphi \circ f)| = |\varphi'| \circ f |Df|$, \quad for $\varphi : \mathbb{R} \to \mathbb{R}$ Lipschitz

Leibniz rule: $|D(fg)| \leq |f||Dg| + |g||Df|$, \quad for $f, g \in S^2 \cap L^\infty(\mathcal{X})$
The ‘Pre-cotangent module’

Consider the set

\[ P_{cm} := \left\{ (A_i, f_i)_{i \in \mathbb{N}} : (A_i) \text{ is a Borel partition of } \mathcal{X} \right. \]
\[ \left. f_i \in S^2(\mathcal{X}) \text{ for every } i \in \mathbb{N} \right\} \]

Define an equivalence relation \( \sim \) on \( P_{cm} \) by declaring \((A_i, f_i)_{i \in \mathbb{N}} \sim (B_j, g_j)_{j \in \mathbb{N}}\) provided for any \( i, j \in \mathbb{N} \) we have \(|D(f_i - g_j)| = 0\) \(m-a.e.\) on \( \{A_i \cap B_j\} \).

Denote by \([A_i, f_i]\) the equivalence class of \((A_i, f_i)_{i \in \mathbb{N}}\).
The ‘Pre-cotangent module’

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\[ f_i \in S^2(\mathcal{X}) \text{ for every } i \in \mathbb{N} \]

\[ \sum_i \int_{A_i} |Df_i|^2 \, dm < \infty \]

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Denote by \([A_i, f_i]\) the equivalence class of \((A_i, f_i)_{i \in \mathbb{N}}\)
Operations on $\text{Pcm}$

Sum

\[ [A_i, f_i] + [B_j, g_j] := [A_i \cap B_j, f_i + g_j] \]

Multiplication by a simple function

For $h = \sum_j \alpha_j \chi_{E_j}$ we put

\[ h \cdot [A_i, f_i] := [A_i \cap E_j, \alpha_j f_i] \]

Pointwise norm

\[ ||[A_i, f_i]| := |Df_i|, \quad m - a.e. \text{ on } A_i \]

Norm

\[ \| [A_i, f_i] \| := \sqrt{\int_{\mathcal{X}} |[A_i, f_i]|^2 \, dm} = \sqrt{\sum_i \int_{A_i} |Df_i|^2 \, dm} \]
The cotangent module $L^2(T^*\mathcal{X})$

We define $L^2(T^*\mathcal{X})$ to be the completion of $(\text{Pcm}/\sim, \|\cdot\|)$. Its elements are called 1-forms.

All the operations can be extended by continuity endowing $L^2(T^*\mathcal{X})$ with the structure of $L^2$-normed $L^\infty$-module.
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For $f \in S^2(\mathcal{X})$ the differential $df \in L^2(T^*\mathcal{X})$ is defined as

$$df := [\mathcal{X}, f]$$
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Note: in the smooth case, the construction is canonically identifiable with the space of $L^2$ sections of the cotangent bundle.
Calculus rules for $df$

Closure:

$$\begin{align*}
(f_n) & \subset S^2(\mathcal{X}) \\
\mathbf{f}_n & \to f \quad \text{m} - \text{a.e.} \\
\mathbf{df}_n & \to \omega \quad \text{in } L^2(T^*\mathcal{X})
\end{align*} \implies \begin{cases}
\{ f \in S^2(\mathcal{X}) \\
\mathbf{df} = \omega
\end{cases}$$

Linearity: $\mathbf{d}(\alpha f + \beta g) = \alpha \mathbf{df} + \beta \mathbf{dg}$

Locality: $\mathbf{df} = \mathbf{dg} \quad \text{m} - \text{a.e. on } \{ f = g \}$

Chain rule: $\mathbf{d}(\varphi \circ f) = \varphi' \circ f \mathbf{df}$, for $\varphi : \mathbb{R} \to \mathbb{R}$ Lipschitz

Leibniz rule: $\mathbf{d}(fg) = f \mathbf{dg} + g \mathbf{df}$, for $f, g \in S^2 \cap L^\infty(\mathcal{X})$
The tangent module

**Definition** The tangent module $L^2(\mathcal{T}X)$ is the dual of the cotangent one. Its elements are called vector fields.
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**Definition** The tangent module $L^2(T\mathcal{X})$ is the dual of the cotangent one. Its elements are called vector fields.

A gradient of $f \in S^2$ is any vector field $X$ such that

$$df(X) = \frac{1}{2}|df|^2 + \frac{1}{2}|X|^2 \quad \text{m} - \text{a.e.}$$
Basic facts about the tangent module

- For any \( f, g \in S^2(X) \) we have

\[
D^+ f(\nabla g) \geq df(X) \quad m\text{-a.e.}
\]

for any \( X \) gradient of \( g \). Equality is realized \( m\)-a.e. for an appropriate choice of gradient.
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- For any $f, g \in S^2(X)$ we have
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- Vector fields are in 1-1 correspondence with $L^2$ derivations, i.e. maps $L : S^2(\mathcal{X}) \to L^1(m)$ satisfying the Leibniz rule and such that
  \[ |L(f)| \leq l|Df| \quad m \text{ - a.e.}, \]
  for some $l \in L^2(m)$. 
Basic facts about the tangent module

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$$|L(f)| \leq l|Df| \quad m - a.e.,$$

for some $l \in L^2(m)$.

For any vector field $X \in L^2(T\mathcal{X})$ we have its pointwise norm $|X| \in L^2(\mathcal{X})$. It can be seen that such norm induces, in an appropriate weak sense, the original distance $d$ on $\mathcal{X}$. 

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Maps of bounded deformation

A map \( \varphi : \mathcal{X}_1 \to \mathcal{X}_2 \) is of bounded deformation provided

\[
\text{Lip}(\varphi) < \infty
\]

\[
\varphi_* m_1 \leq C m_2, \quad \text{for some } C > 0
\]
Maps of bounded deformation

A map $\varphi : X_1 \to X_2$ is of bounded deformation provided

$$\text{Lip}(\varphi) < \infty$$
$$\varphi_* m_1 \leq C m_2, \quad \text{for some } C > 0$$

It can be proved that for $f \in S^2(X_2)$ we have $f \circ \varphi \in S^2(X_1)$ with

$$|D(\varphi \circ f)| \leq \text{Lip}(\varphi)|Df| \circ \varphi$$
Theorem Let $\varphi : \mathcal{X}_1 \to \mathcal{X}_2$ be of bounded deformation. Then there exists a unique linear continuous map $\varphi^* : L^2(T^*\mathcal{X}_2) \to L^2(T^*\mathcal{X}_1)$ such that

$$
\varphi^* df = d(f \circ \varphi) \\
\varphi^*(g\omega) = g \circ \varphi \varphi^* \omega
$$

Such map satisfies

$$
|\varphi^* \omega| \leq \text{Lip}(\varphi)|\omega| \circ \varphi, \quad m_1 - a.e..
$$
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3 simple formulas

\[ Hf(\nabla g, \nabla g) = \langle \nabla \langle \nabla f, \nabla g \rangle, \nabla g \rangle - \frac{1}{2} \langle \nabla f, \nabla |\nabla g|^2 \rangle \]

\[ \langle \nabla_{\nabla f} X, \nabla g \rangle = \langle \nabla \langle X, \nabla g \rangle, \nabla f \rangle - Hg(X, \nabla f) \]

\[ d\omega(X, Y) = X(\omega(Y)) - X(\omega(Y)) - \omega([X, Y]) \]
Why these can be used on $\text{RCD}(K, \infty)$ spaces

On $\text{RCD}(K, \infty)$ spaces we have

$$\Delta \frac{|\nabla f|^2}{2} \geq \langle \nabla f, \nabla \Delta f \rangle + K |\nabla f|^2$$

in the weak sense.
Why these can be used on RCD($K, \infty$) spaces

On $RCD(K, \infty)$ spaces we have

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in the weak sense. In the smooth setting this actually implies

$$\Delta \frac{|\nabla f|^2}{2} \geq |Hf|_{HS}^2 + \langle \nabla f, \nabla \Delta f \rangle + K|\nabla f|^2$$
Why these can be used on RCD\((K, \infty)\) spaces

On \(RCD(K, \infty)\) spaces we have

\[
\Delta \frac{\abs{\nabla f}^2}{2} \geq \langle \nabla f, \nabla \Delta f \rangle + K \abs{\nabla f}^2
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in the weak sense. In the smooth setting this actually implies

\[
\Delta \frac{\abs{\nabla f}^2}{2} \geq \|Hf\|_{HS}^2 + \langle \nabla f, \nabla \Delta f \rangle + K \abs{\nabla f}^2
\]

The passage from the former to the latter can be obtained by purely algebraic means (Bakry ’85, Savaré ’12, Sturm ’14, G. ’14) and in particular the latter holds in \(RCD(K, \infty)\) spaces.
Why these can be used on $RCD(K, \infty)$ spaces

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$$\Delta \frac{|\nabla f|^2}{2} \geq |Hf|^2_{\text{HS}} + \langle \nabla f, \nabla \Delta f \rangle + K|\nabla f|^2$$

The passage from the former to the latter can be obtained by purely algebraic means (Bakry '85, Savaré '12, Sturm '14, G. '14) and in particular the latter holds in $RCD(K, \infty)$ spaces. Integrating we get:

$$\int |Hf|^2_{\text{HS}} \, dm \leq \int |\Delta f|^2 - K|\nabla f|^2 \, dm$$
Where this brings

Definition of the Sobolev space $W^{2,2}(X)$ and of the Hessian.
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Definition of the Sobolev space $W_{C}^{1,2}(T\mathcal{X})$ and of the covariant derivative, which is compatible with the metric and torsion free in a natural way.
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Definition of the Sobolev space $W^{1,2}_{d}(T^*\mathcal{X})$ and of exterior differential, which leads to de Rham cohomology and Hodge theory.
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Definition of the Ricci curvature via the formula

$$Ric(X, X) := \Delta \frac{|X|^2}{2} - |\nabla X|_{HS}^2 + \langle X, \Delta_H X \rangle$$

which is a measure-valued operator satisfying

$$Ric(X, X) \geq K|X|^2 m$$
In summary:

Spaces with Ricci curvature bounded from below by $K$ have Ricci curvature bounded from below by $K$. 
In summary:

Spaces with Ricci curvature bounded from below by $K$ have Ricci curvature bounded from below by $K$
About rectifiability

**Theorem** (Mondino-Naber ’14 / in progress) Let \((X, d, m)\) be a RCD\((K, N)\) space and \(\varepsilon > 0\).
Then there is a Borel partition \((A_n)\) of \(X\) and maps \(\varphi_n : A_n \to \mathbb{R}^{d_n}\) with \(d_n \leq N\) such that
\[
\text{Lip}(\varphi_n), \text{Lip}(\varphi_n^{-1}) \leq 1 + \varepsilon
\]
and putting \(\mu_n := (\varphi_n)_*(m|_{A_n})\) we have \(\mu_n = \rho_n \mathcal{L}^{d_n}\) with
\[
\text{ess-sup}_{\varphi_n(A_n)} \rho_n - \text{ess-inf}_{\varphi_n(A_n)} \rho_n \leq \varepsilon
\]
Theorem (Mondino-Naber ’14 / in progress) Let \((\mathcal{X}, d, m)\) be a RCD\((K, N)\) space and \(\varepsilon > 0\).
Then there is a Borel partition \((A_n)\) of \(\mathcal{X}\) and maps \(\varphi_n : A_n \to \mathbb{R}^{d_n}\) with \(d_n \leq N\) such that
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\[
\text{ess-sup} \rho_n - \text{ess-inf} \rho_n \leq \varepsilon
\]
\(\varphi_n(A_n)\) \(\varphi_n(A_n)\)

In particular, recalling the properties of the pullback of 1-forms we get:

Corollary The tangent module \(L^2(T\mathcal{X})\) is canonically isomorphic to the space of Borel and \(L^2\) maps assigning to \(m\)-a.e. \(x \in \mathcal{X}\) an element of the pmGH-limit of rescaled spaces.