Spaces with Ricci curvature bounded from below

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Topics

1) On the definition of spaces with Ricci curvature bounded from below

2) Analytic properties of $RCD^*(K, N)$ spaces

3) Geometric properties of $RCD^*(K, N)$ spaces

4 More on the differential structure of metric measure spaces

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- First order differential structure of metric measure spaces
  - L^2-normed L^∞-modules
  - The construction
  - Behavior under transformations
- Second order differential structure of RCD spaces

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The setting

$(\mathcal{X}, \mathsf{d}, \mathfrak{m})$ is such that:

- $(\mathcal{X}, \mathsf{d})$ is complete and separable
- $\mathfrak{m}$ is a non-negative Radon measure on it

L^2-normed $L^\infty(\mathfrak{m})$-modules

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- a multiplication with $L^\infty(\mathfrak{m})$ functions, i.e. a bilinear map $L^\infty(\mathfrak{m}) \times M \to M$ satisfying

$$f(gv) = (fg)v,$$
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for every $f, g \in L^\infty(\mathfrak{m})$ and $v \in M$.

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  for every $f, g \in L^\infty(\mathfrak{m})$ and $v \in M$.
- a pointwise L^2-norm, i.e. a map $|\cdot| : M \to L^2(\mathfrak{m})$ satisfying

$$|v| \geq 0, \qquad \mathfrak{m} - a.e.,$$
$$|fv| = |f|\,|v|, \qquad \mathfrak{m} - a.e.,$$
$$\|v\|_M = \sqrt{\int |v|^2 \, \mathrm{d}\mathfrak{m}}$$

Why such a notion

Basic example:
The space of L^2 vector fields on a Riemannian manifold.

More generally, the space of L^2 sections of a normed vector bundle.

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The idea:
In the smooth setting, one can fully describe a vector bundle by looking either to its fibers, or to its sections.

In the non-smooth one, we take the latter viewpoint and thus declare L^2-normed $L^\infty(\mathfrak{m})$-modules to 'be' vector bundles on our metric measure space

Basic features of modules: locality

For $v, w \in M$ and a Borel set $E \subset \mathcal{X}$ we say that

$$v = w, \qquad \mathfrak{m} - a.e.\ on\ E$$

provided

$$\chi_E(v - w) = 0.$$

or equivalently

$$|v - w| = 0, \qquad \mathfrak{m} - a.e.\ on\ E.$$

Basic features of modules: duality

The dual M^* of M is the space of linear continuous maps $L : M \to L^1(\mathfrak{m})$ which are local, i.e. such that

$$L(fv) = f\,L(v), \qquad \forall v \in M,\ f \in L^\infty(\mathfrak{m}).$$

M^* is also an L^2-normed L^∞-module, the pointwise norm being given by

$$|L|_* := \underset{v\,:\,|v|\leq 1\ \mathfrak{m}-a.e.}{\text{ess-sup}} L(v)$$

Example: the dual of $L^2(\mathfrak{m})$

The dual of $L^2(\mathfrak{m})$ as Hilbert space is $L^2(\mathfrak{m})$, i.e. for $L : L^2(\mathfrak{m}) \to \mathbb{R}$ linear and continuous there is a unique $g \in L^2(\mathfrak{m})$ such that

$$L(f) = \int fg \, \mathrm{d}\mathfrak{m} \qquad \forall f \in L^2(\mathfrak{m})$$

and $\|L\|_{L^2(\mathfrak{m})'} = \|g\|_{L^2(\mathfrak{m})}$. And viceversa.

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The dual of $L^2(\mathfrak{m})$ as module is $L^2(\mathfrak{m})$, i.e. for $T : L^2(\mathfrak{m}) \to L^1(\mathfrak{m})$ linear, continuous and local, there is a unique $g \in L^2(\mathfrak{m})$ such that

$$T(f) = fg \quad \mathfrak{m} - a.e. \qquad \forall f \in L^2(\mathfrak{m})$$

and $|T|_* = |g|$ $\mathfrak{m}$-a.e.. And viceversa.

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Calculus rules for $|Df|$

Lower semicontinuity:

$$\left.\begin{array}{rl} (f_n) & \subset S^2(\mathcal{X}) \\ f_n & \to f \qquad \mathfrak{m}-a.e. \\ |Df_n| & \rightharpoonup G \text{ in } L^2(\mathfrak{m}) \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{rl} f & \in S^2(\mathcal{X}) \\ |Df| & \le G \end{array}\right.$$

Subadditivity: $|D(\alpha f+\beta g)| \le |\alpha||Df| + |\beta||Dg| \qquad \mathfrak{m}-a.e.$

Locality: $|Df| = |Dg| \qquad \mathfrak{m}-a.e.\ on\ \{f=g\}$

Chain rule: $|D(\varphi\circ f)| = |\varphi'|\circ f\,|Df|,$ for $\varphi : \mathbb{R}\to\mathbb{R}$ Lipschitz

Leibniz rule: $|D(fg)| \le |f||Dg| + |g||Df|,$ for $f, g \in S^2 \cap L^\infty(\mathcal{X})$

The 'Pre-cotangent module'

Consider the set

$$\mathrm{Pcm} := \Big\{(A_i, f_i)_{i\in\mathbb{N}} \ : (A_i) \text{ is a Borel partition of } \mathcal{X}$$
$$f_i \in S^2(\mathcal{X}) \text{ for every } i \in \mathbb{N}$$
$$\sum_i \int_{A_i} |Df_i|^2 \,\mathrm{d}\mathfrak{m} < \infty \Big\}$$

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Define an equivalence relation $\sim$ on Pcm by declaring

$$(A_i, f_i)_{i\in\mathbb{N}} \sim (B_j, g_j)_{j\in\mathbb{N}}$$

provided for any $i, j \in \mathbb{N}$ we have

$$|D(f_i - g_j)| = 0 \quad \mathfrak{m} - a.e. \text{ on } \{A_i \cap B_j\}$$

Denote by $[A_i, f_i]$ the equivalence class of $(A_i, f_i)_{i\in\mathbb{N}}$

Operations on $\mathrm{Pcm}/\sim$

Sum

$$[A_i, f_i] + [B_j, g_j] := [A_i \cap B_j, f_i + g_j]$$

Multiplication by a simple function For $h = \sum_j \alpha_j \chi_{E_j}$ we put

$$h \cdot [A_i, f_i] := [A_i \cap E_j, \alpha_j f_i]$$

Pointwise norm

$$|[A_i, f_i]| := |Df_i|, \qquad \mathfrak{m} - a.e.\ on\ A_i$$

Norm

$$\|[A_i, f_i]\| := \sqrt{\int_{\mathcal{X}} |[A_i, f_i]|^2 \,\mathrm{d}\mathfrak{m}} = \sqrt{\sum_i \int_{A_i} |Df_i|^2 \,\mathrm{d}\mathfrak{m}}$$

The cotangent module $L^2(T^*\mathcal{X})$

We define $L^2(T^*\mathcal{X})$ to be the completion of $(\mathrm{Pcm}/\sim, \|\cdot\|)$. Its elements are called 1-forms.

All the operations can be extended by continuity endowing $L^2(T^*\mathcal{X})$ with the structure of L^2-normed L^∞-module.

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Note: in the smooth case, the construction is canonically identifiable with the space of L^2 sections of the cotangent bundle.

Calculus rules for $\mathrm{d}f$

Closure:

$$\left.\begin{array}{rl} (f_n) & \subset S^2(\mathcal{X}) \\ f_n & \to f \qquad \mathfrak{m}-a.e. \\ \mathrm{d}f_n & \rightharpoonup \omega \ \text{ in } L^2(T^*\mathcal{X}) \end{array}\right\} \qquad \Rightarrow \qquad \left\{\begin{array}{rl} f & \in S^2(\mathcal{X}) \\ \mathrm{d}f & = \omega \end{array}\right.$$

Linearity: $\mathrm{d}(\alpha f + \beta g) = \alpha \mathrm{d}f + \beta \mathrm{d}g$

Locality: $\mathrm{d}f = \mathrm{d}g \qquad \mathfrak{m} - a.e.\ on\ \{f = g\}$

Chain rule: $\mathrm{d}(\varphi \circ f) = \varphi' \circ f\, \mathrm{d}f, \qquad$ for $\varphi : \mathbb{R} \to \mathbb{R}$ Lipschitz

Leibniz rule: $\mathrm{d}(fg) = f\, \mathrm{d}g + g\, \mathrm{d}f, \qquad$ for $f, g \in S^2 \cap L^\infty(\mathcal{X})$

The tangent module

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A gradient of $f \in S^2$ is any vector field X such that

$$\mathrm{d}f(X) = \frac{1}{2}|\mathrm{d}f|^2 + \frac{1}{2}|X|^2 \qquad \mathfrak{m} - a.e..$$

Basic facts about the tangent module

- For any $f, g \in S^2(X)$ we have

$$D^+ f(\nabla g) \geq \mathrm{d}f(X) \qquad \mathfrak{m} - a.e.$$

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- Vector fields are in 1-1 correspondence with L^2 derivations, i.e. maps $L : S^2(\mathcal{X}) \to L^1(\mathfrak{m})$ satisfying the Leibniz rule and such that

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- For any vector field $X \in L^2(T\mathcal{X})$ we have its pointwise norm $|X| \in L^2(\mathcal{X})$. It can be seen that such norm induces, in an appropriate weak sense, the original distance d on $\mathcal{X}$.

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Maps of bounded deformation

A map $\varphi : \mathcal{X}_1 \to \mathcal{X}_2$ is of bounded deformation provided

$$\begin{aligned} &\mathrm{Lip}(\varphi) < \infty \\ &\varphi_* \mathfrak{m}_1 \le C \mathfrak{m}_2, \qquad \text{for some } C > 0 \end{aligned}$$

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It can be proved that for $f \in S^2(\mathcal{X}_2)$ we have $f \circ \varphi \in S^2(\mathcal{X}_1)$ with

$$|D(\varphi \circ f)| \le \mathrm{Lip}(\varphi)|Df| \circ \varphi$$

Pullback of 1-forms

Theorem Let $\varphi : \mathcal{X}_1 \to \mathcal{X}_2$ be of bounded deformation. Then there exists a unique linear continuous map $\varphi^* : L^2(T^*\mathcal{X}_2) \to L^2(T^*\mathcal{X}_1)$ such that

$$\varphi^* \mathrm{d}f = \mathrm{d}(f \circ \varphi)$$
$$\varphi^*(g\omega) = g \circ \varphi \, \varphi^* \omega$$

Such map satisfies

$$|\varphi^* \omega| \leq \mathrm{Lip}(\varphi) |\omega| \circ \varphi, \qquad \mathfrak{m}_1 - a.e..$$

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3 simple formulas

$$\mathrm{H}f(\nabla g, \nabla g) = \langle \nabla \langle \nabla f, \nabla g \rangle, \nabla g \rangle - \tfrac{1}{2} \left\langle \nabla f, \nabla |\nabla g|^2 \right\rangle$$

$$\langle \nabla_{\nabla f} X, \nabla g \rangle = \langle \nabla \langle X, \nabla g \rangle, \nabla f \rangle - \mathrm{H}g(X, \nabla f)$$

$$\mathrm{d}\omega(X, Y) = X\big(\omega(Y)\big) - X\big(\omega(Y)\big) - \omega([X, Y])$$

Why these can be used on RCD(K, ∞) spaces

On $RCD(K, \infty)$ spaces we have

$$\Delta \frac{|\nabla f|^2}{2} \geq \langle \nabla f, \nabla \Delta f \rangle + K |\nabla f|^2$$

in the weak sense.

Why these can be used on RCD(K, ∞) spaces

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Integrating we get:

$$\int|\mathrm{H}f|^2_{\mathsf{HS}}\,\mathrm{d}\mathfrak{m}\leq\int|\Delta f|^2-K|\nabla f|^2\,\mathrm{d}\mathfrak{m}$$

Where this brings

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Definition of the Ricci curvature via the formula

$$\mathrm{Ric}(X,X) := \Delta\frac{|X|^2}{2} - |\nabla X|^2_{\mathsf{HS}} + \langle X, \Delta_{\mathsf{H}} X\rangle$$

which is a measure-valued operator satisfying

$$\mathrm{Ric}(X,X) \geq K|X|^2\mathfrak{m}$$

In summary:

Spaces with Ricci curvature bounded from below by K have Ricci curvature bounded from below by K

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Thank you

About rectifiability

Theorem (Mondino-Naber '14 / in progress) Let $(\mathcal{X}, \mathsf{d}, \mathfrak{m})$ be a $\mathsf{RCD}(K, N)$ space and $\varepsilon > 0$.
Then there is a Borel partition (A_n) of $\mathcal{X}$ and maps $\varphi_n : A_n \to \mathbb{R}^{d_n}$ with $d_n \leq N$ such that

$$\mathrm{Lip}(\varphi_n),\ \mathrm{Lip}(\varphi_n^{-1}) \leq 1 + \varepsilon$$

and putting $\mu_n := (\varphi_n)_*(\mathfrak{m}|_{A_n})$ we have $\mu_n = \rho_n \mathcal{L}^{d_n}$ with

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In particular, recalling the properties of the pullback of 1-forms we get:

Corollary The tangent module $L^2(T\mathcal{X})$ is canonically isomorphic to the space of Borel and L^2 maps assigning to $\mathfrak{m}$-a.e. $x \in \mathcal{X}$ an element of the pmGH-limit of rescaled spaces.