Spaces with Ricci curvature bounded from below

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Topics

1) On the definition of spaces with Ricci curvature bounded from below

2) Analytic properties of $RCD(K, N)$ spaces

3) Geometric properties of $RCD(K, N)$ spaces

4) More on the differential structure of metric measure spaces

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3) Geometric properties of $RCD(K,N)$ spaces

4) More on the differential structure of metric measure spaces

On the definition of spaces with Ricci curvature bounded from below

- Introduction
- The gradient flow of the relative entropy w.r.t. W_2
- The gradient flow of the Dirichlet energy w.r.t. L^2
- The heat flow as gradient flow

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Completion / Compactification

A common practice in various fields of mathematic is to start studying a certain class of ‘smooth’ or ‘nice’ objects, and to close it w.r.t. some relevant topology.

In general, the study of the limit objects turns out to be useful to understand the properties of the original ones.

Gromov's plan

When the original class of objects are Riemannian manifolds with some curvature bounds, this program has been proposed by Gromov.

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Bounds from below on the Ricci curvature	measured Gromov-Hausdorff convergence	$CD(K,N)$ spaces / $RCD(K,N)$ spaces

Aim of the game

(1) To understand what it means for a metric measure space to have Ricci curvature bounded from below

(2) To prove in the non-smooth setting ‘all’ the theorems valid for manifolds with $\mathrm{Ric} \geq K$, $\dim \leq N$ and their limits

(3) To better understand the geometry of smooth manifolds via the study of non-smooth objects

The curvature condition

Theorem (Sturm-VonRenesse '05) *- see also Otto-Villani and Cordero Erausquin-McCann-Schmuckenschlager*
Let M be a smooth Riemannian manifold. Then the following are equivalent:

i) The Ricci curvature of M is uniformly bounded from below by K
ii) The relative entropy functional is K-convex on the space $(\mathscr{P}_2(M), W_2)$

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Basic features of the $CD(K, \infty)$ condition:

- Compatibility with the Riemannian case
- Stability w.r.t. mG convergence
- More general $CD(K, N)$ spaces can be introduced

Finsler structures are included

Cordero-Erausquin, Villani, Sturm proved that $(\mathbb{R}^d, \|\cdot\|, \mathcal{L}^d)$ is a $CD(0,\infty)$ space (in fact $CD(0,d)$) for any norm.

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Some differences between the Finsler and Riemannian worlds:

Analysis:	Geometry:
Tangent / cotangent spaces can't be identified	no Abresch-Gromoll inequality
No natural Dirichlet form	no Splitting theorem

Some observations

- For a given Finsler manifold the following are equivalent:
  - The manifold is Riemannian
  - The Sobolev space $W^{1,2}$ is Hilbert
  - The heat flow is linear

- The heat flow can be seen as:
  - Gradient flow of the Dirichlet energy w.r.t. L^2
  - Gradient flow of the relative entropy w.r.t. W_2

The idea

Restrict to the class of $CD(K,\infty)$ spaces such that $W^{1,2}$ is Hilbert, because *'if we were able to make computations as in Riemannian manifolds, we should be able to prove the same results'*.

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- noticing that, even in the non-smooth case, $W^{1,2}$ is Hilbert if and only if the L^2 gradient flow of the Dirichlet energy is linear
- proving that the L^2 gradient flow of the Dirichlet energy coincides with the W_2-gradient flow of the relative entropy
- proving that the W_2-gradient flow of the relative entropy is stable

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Plan pursued in:
G. '09
G., Kuwada, Ohta '10
Ambrosio, G., Savaré '11
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Definition of Gradient Flow: the smooth case

Let $(x_t) \subset \mathbb{R}^d$ a smooth curve and $f : \mathbb{R}^d \to \mathbb{R}$ a smooth functional. Then

$$\begin{aligned} f(x_0) - f(x_t) &\leq \int_0^t |x_s'||\nabla f|(x_s)\,\mathrm{d}s \\ &\leq \frac{1}{2}\int_0^t |x_s'|^2 + |\nabla f|^2(x_s)\,\mathrm{d}s. \end{aligned}$$

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\end{aligned}
$$

Therefore

$$
\begin{gathered}
x_t' = -\nabla f(x_t), \qquad \forall t \geq 0 \\
\Updownarrow \\
f(x_0) = f(x_t) + \frac{1}{2} \int_0^t |x_s'|^2 + |\nabla f|^2(x_s) \,\mathrm{d}s, \quad \forall t > 0.
\end{gathered}
$$

Definition of Gradient Flow: the metric setting

- $|\dot{x}_t| := \lim_{h \to 0} \dfrac{d(x_{t+h}, x_t)}{|h|}$ for an abs.cont. curve (x_t)
- $|\partial^- F|(x) := \overline{\lim_{y \to x}} \dfrac{(F(x) - F(y))^+}{d(x, y)}$
- The weak chain rule

$$F(x_0) \leq F(x_t) + \frac{1}{2} \int_0^t |\dot{x}_s|^2 + |\partial^- F|^2(x_s) \, \mathrm{d}s, \qquad \forall t > 0.$$

holds for K-convex and l.s.c. $F : X \to \mathbb{R} \cup \{+\infty\}$.

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Definition (x_t) is a Gradient Flow for the K-conv. and l.s.c. functional F provided $(x_t) \subset \{F < \infty\}$ is a loc.abs.cont. curve and

$$F(x_0) = F(x_t) + \frac{1}{2}\int_0^t |\dot{x}_s|^2 + |\partial^- F|^2(x_s)\,\mathrm{d}s, \quad \forall t > 0.$$

General results about GF of K-convex functionals

Existence Granted if the space is compact and $F(x_0) < \infty$ (Ambrosio, G., Savaré '04 (after De Giorgi))

Uniqueness False in general

Basic facts about the GF of the Entropy

Thm. (G. '09)
Let $(X, \mathsf{d}, \mathfrak{m})$ be a compact $CD(K, \infty)$ space.

Then for $\mu \in \mathscr{P}_2(X)$ with $\mathrm{Ent}_{\mathfrak{m}}(\mu) < \infty$ the GF of $\mathrm{Ent}_{\mathfrak{m}}$ starting from μ exists and is unique.

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Furthermore, such flow is stable w.r.t. mG-convergence of the base space.

Proof of uniqueness

Key Lemma Let $(X, \mathsf{d}, \mathfrak{m})$ be a compact $CD(K, \infty)$ space. Then $|\partial^- \mathrm{Ent}_{\mathfrak{m}}|^2(\cdot)$ is convex w.r.t. linear interpolation.

Proof of uniqueness

Key Lemma Let $(X, \mathsf{d}, \mathfrak{m})$ be a compact $CD(K, \infty)$ space. Then $|\partial^- \mathrm{Ent}_{\mathfrak{m}}|^2(\cdot)$ is convex w.r.t. linear interpolation.

Then by contradiction: assume $(\mu_t), (\nu_t)$ are two GF starting from $\bar{\mu}$ with $\mathrm{Ent}_{\mathfrak{m}}(\bar{\mu}) < \infty$ and define $\sigma_t := \frac{\mu_t + \nu_t}{2}$.

Then for every t such that $\mu_t \neq \nu_t$ we have

$$\mathrm{Ent}_{\mathfrak{m}}(\bar{\mu}) > \mathrm{Ent}_{\mathfrak{m}}(\sigma_t) + \frac{1}{2}\int_0^t |\dot{\sigma}_s|^2 + |\partial^- \mathrm{Ent}_{\mathfrak{m}}|^2(\sigma_s)\,\mathrm{d}s$$

mG convergence of compact spaces

$(X_n, \mathsf{d}_n, \mathfrak{m}_n)$ converges to $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty)$ in the mG sense if there is (Y, d_Y) and isometric embeddings ι_n, ι_∞ of the X's into Y such that

$$(\iota_n)_\sharp \mathfrak{m}_n \qquad \text{weakly converges to} \qquad (\iota_\infty)_\sharp \mathfrak{m}_\infty$$

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We say that $n \mapsto \mu_n \in \mathscr{P}(X_n)$ weakly converges to $\mu_\infty \in \mathscr{P}(X_\infty)$ provided

$$(\iota_n)_\sharp \mu_n \qquad \text{weakly converges to} \qquad (\iota_\infty)_\sharp \mu_\infty$$

Γ-convergence of the entropies

Thm. (Lott-Sturm-Villani)
Let $(X_n, \mathsf{d}_n, \mathfrak{m}_n)$ be converging to $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty)$. Then:

- $\Gamma - \underline{\lim}$ inequality: for every sequence $n \mapsto \mu_n \in \mathscr{P}(X_n)$ weakly converging to $\mu_\infty \mathscr{P}(X_\infty)$ we have

$$\mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_\infty) \leq \varliminf_{n\to\infty} \mathrm{Ent}_{\mathfrak{m}_n}(\mu_n).$$

- $\Gamma - \overline{\lim}$ inequality: for every $\mu_\infty \in \mathscr{P}(X_\infty)$ there is a sequence $n \mapsto \mu_n \in \mathscr{P}(X_n)$ weakly converging to μ_∞ such that

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Cor. The $CD(K, \infty)$ condition is closed w.r.t. mG convergence.

$\Gamma - \underline{\lim}$ for the slopes

Thm. (G. '09) Let X_n be $CD(K,\infty)$ spaces mG-converging to X_∞ and μ_n weakly converging to μ_∞. Then

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Cor. 1 Let X_n be mG-converging to X_∞ and μ_n be weakly converging to μ_∞ be such that

$$\lim_{n\to\infty} \mathrm{Ent}_{\mathfrak{m}_n}(\mu_n) = \mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_\infty) < \infty.$$

Then the GF of $\mathrm{Ent}_{\mathfrak{m}_n}$ starting from μ_n converge to the GF of $\mathrm{Ent}_{\mathfrak{m}_\infty}$ starting from μ_∞.

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Then the GF of $\mathrm{Ent}_{\mathfrak{m}_n}$ starting from μ_n converge to the GF of $\mathrm{Ent}_{\mathfrak{m}_\infty}$ starting from μ_∞.

Cor. 2 The condition '$CD(K,\infty)$+linearity of the GF of the entropy' is closed w.r.t. mG convergence.

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Variational definition of $|Df|$ on $\mathbb{R}^d$

Let $f : \mathbb{R}^d \to \mathbb{R}$ be smooth.

Then $|Df|$ is the minimum continuous function G for which

$$|f(\gamma_1) - f(\gamma_0)| \leq \int_0^1 G(\gamma_t)|\dot{\gamma}_t| \, \mathrm{d}t$$

holds for any smooth curve γ

Test plans

Let $\boldsymbol{\pi} \in \mathscr{P}(C([0,1],X))$. We say that $\boldsymbol{\pi}$ is a test plan provided:

- for some $C > 0$ it holds

$$\mathrm{e}_{t\,\sharp}\boldsymbol{\pi} \leq C\mathfrak{m}, \qquad \forall t \in [0,1].$$

- it holds

$$\iint_0^1 |\dot{\gamma}_t|^2 \,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi} < \infty$$

The Sobolev class $S^2(X, \mathsf{d}, \mathfrak{m})$

We say that $f : X \to \mathbb{R}$ belongs to $S^2(X, \mathsf{d}, \mathfrak{m})$ provided there exists $G \in L^2(X, \mathfrak{m})$, $G \geq 0$ such that

$$\int \big|f(\gamma_1) - f(\gamma_0)\big| \, \mathrm{d}\boldsymbol{\pi}(\gamma) \leq \iint_0^1 G(\gamma_t)|\dot\gamma_t| \, \mathrm{d}t \, \mathrm{d}\boldsymbol{\pi}(\gamma)$$

for any test plan $\boldsymbol{\pi}$.

Any such G is called weak upper gradient of f.

The minimal G in the $\mathfrak{m}$-a.e. sense will be denoted by $|Df|$

Basic properties

Lower semicontinuity From $f_n \to f$ $\mathfrak{m}$-a.e. with $f_n \in S^2$ and $|Df_n| \to G$ weakly in L^2 we deduce

$$f \in S^2, \qquad |Df| \le G$$

Locality

$$|Df| = |Dg| \qquad \mathfrak{m}\text{-}a.e.\ on\ \{f = g\}$$

Chain rule

$$|D(\varphi \circ f)| = |\varphi'| \circ f\, |Df|$$

for φ Lipschitz

'Leibniz rule'

$$|D(fg)| \le |f||Dg| + |g||Df|$$

for $f, g \in S^2 \cap L^\infty$

The Energy E and the Sobolev space $W^{1,2}$

We define $E : L^2(X, \mathfrak{m}) \to [0, +\infty]$ as

$$E(f) := \frac{1}{2} \int |Df|^2 \, \mathrm{d}\mathfrak{m} \quad \text{if } f \in S^2(X), \qquad +\infty \quad \text{otherwise.}$$

Then E is convex and lower semicontinuous.

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Then E is convex and lower semicontinuous.

The Sobolev space $W^{1,2}(X)$ is $W^{1,2}(X) := L^2(X) \cap S^2(X)$ endowed with the norm

$$\|f\|^2_{W^{1,2}} := \|f\|^2_{L^2} + \||Df|\|^2_{L^2}$$

$W^{1,2}(X)$ is a Banach space.

Laplacian (first definition)

We say that $f \in D(\Delta) \subset W^{1,2}(X)$ if $\partial^- E(f) \neq 0$.

In this case we define $\Delta f := -v$, where v is the element of minimal norm in $\partial^- E(f)$.

‘Integration by parts’

For $f \in D(\Delta)$ and $g \in W^{1,2}(X)$ we have

$$\left| \int g \Delta f \, \mathrm{d}\mathfrak{m} \right| \leq \int |Dg||Df| \, \mathrm{d}\mathfrak{m}.$$

For a C^1 map $u : \mathbb{R} \to \mathbb{R}$ we have

$$\int u(f) \Delta f \, \mathrm{d}\mathfrak{m} = - \int u'(f) |Df|^2 \, \mathrm{d}\mathfrak{m}.$$

Gradient flow of E w.r.t. L^2

For any $f_0 \in L^2(X, \mathfrak{m})$ there exists a unique map $t \mapsto f_t \in L^2(X, \mathfrak{m})$ such that

$$\frac{\mathrm{d}^+}{\mathrm{d}t} f_t = \Delta f_t, \qquad \forall t > 0.$$

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The result

Thm. (G., Kuwada, Ohta '10. Ambrosio, G. Savaré '11)
Let $(X, \mathsf{d}, \mathfrak{m})$ be a $CD(K, \infty)$ space and $\mu = f\mathfrak{m} \in \mathscr{P}_2(X)$ with $f \in L^2(X, \mathfrak{m})$. Let

- $t \mapsto f_t$ be the GF of E w.r.t. L^2 starting from f
- $t \mapsto \mu_t$ be the GF of $\mathrm{Ent}_{\mathfrak{m}}$ w.r.t. W_2 starting from μ

Then

$$\mu_t = f_t\mathfrak{m} \qquad \forall t \geq 0.$$

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The idea is to pick a Gradient Flow of E and prove that it is a Gradient Flow of $\mathrm{Ent}_{\mathfrak{m}}$

Main steps of the proof

Let $t \mapsto f_t$ be a Gradient Flow of E starting from a probability density f_0 and define $\mu_t = f_t \mathfrak{m}$.

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We will prove:

- Mass preservation and maximum/minimum principle to show that $c \leq f_0 \leq C$ implies $c \leq f_t \leq C$ for every $t \geq 0$
- That $t \mapsto \mathrm{Ent}_{\mathfrak{m}}(\mu_t)$ is a.c. and

$$-\partial_t \mathrm{Ent}_{\mathfrak{m}}(\mu_t) = \int \frac{|Df_t|^2}{f_t} \, \mathrm{d}\mathfrak{m}$$

- The slope estimate:

$$|\partial^- \mathrm{Ent}_{\mathfrak{m}}|^2(\mu_t) \leq \int \frac{|Df_t|^2}{f_t} \, \mathrm{d}\mathfrak{m}$$

- That (μ_t) is a.c. w.r.t. W_2 and

$$|\dot{\mu}_t|^2 \leq \int \frac{|Df_t|^2}{f_t} \, \mathrm{d}\mathfrak{m}$$

Maximum/minimum principle ad mass preservation

We claim that for any $f \in L^2$ such that $f \geq c$ and any $\tau > 0$ the minimum of

$$g \quad \mapsto \quad \frac{1}{2}\int |Dg|^2 \, \mathrm{d}\mathfrak{m} + \frac{\|f - g\|_{L^2}^2}{2\tau},$$

satisfies $g \geq c$.

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Similarly the minimizer g satisfies $\int g \, \mathrm{d}\mathfrak{m} = \int f \, \mathrm{d}\mathfrak{m}$

Entropy dissipation

For $0 < c \leq f_0 \leq C < \infty$ the curve $t \mapsto f_t \in L^2(X, \mathfrak{m})$ is AC.

Entropy dissipation

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Computing we have

$$\partial_t \int u(f_t) \, \mathrm{d}\mathfrak{m} = \int u'(f_t) \Delta f_t \, \mathrm{d}\mathfrak{m} = -\int u''(f_t) |Df_t|^2 \, \mathrm{d}\mathfrak{m}$$

Slope of the entropy and Fisher information

Lemma (Lott-Villani) For $\mu = f\mathfrak{m}$ with f Lipschitz we have

$$|\partial^- \mathrm{Ent}_{\mathfrak{m}}|^2(\mu) \leq \int \frac{\mathrm{lip}^2 f}{f} \, \mathrm{d}\mathfrak{m}$$

where

$$\mathrm{lip}\, f(x) := \overline{\lim_{y \to x}} \frac{|f(y) - f(x)|}{\mathsf{d}(x, y)}$$

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By relaxation, using the weak lower semicontinuity of the slope we deduce

$$|\partial^- \mathrm{Ent}_{\mathfrak{m}}|^2(\mu) \le \int \frac{|Df|^2}{f}\, \mathrm{d}\mathfrak{m} = 4 \int |D\sqrt{f}|^2\, \mathrm{d}\mathfrak{m}$$

for every $\mu = f\mathfrak{m}$ with $\sqrt{f} \in W^{1,2}$.

The non-trivial property: Kuwada's lemma

Suppose that $\mu_0 := f\mathfrak{m}$ is in $\mathscr{P}_2(X)$.

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Suppose that $\mu_0 := f\mathfrak{m}$ is in $\mathscr{P}_2(X)$.

Then $\mu_t := f_t\mathfrak{m}$ is in $\mathscr{P}_2(X)$ and the curve $t \mapsto \mu_t$ is absolutely continuous w.r.t. W_2 and

$$|\dot{\mu}_t|^2 \leq \int \frac{|Df_t|^2}{f_t}\,\mathrm{d}\mathfrak{m}$$

Hamilton-Jacobi semigroup alias Hopf-Lax formula alias Moreau-Yosida approximation alias inf-convolution

Let (X, d) be a metric space.
For $\psi : X \to \mathbb{R}$ Lipschitz and bounded $t > 0$ we define $Q_t\psi : X \to \mathbb{R}$ by

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Then:

- $Q_t\psi$ is Lipschitz for every $t > 0$,
- $t \mapsto Q_t\psi$ is a Lipschitz curve w.r.t. the sup distance ($Q_0\psi := \psi$),

Solutions of Hamilton-Jacobi equation

For any $x \in X$ the map $t \mapsto Q_t\varphi(x)$ is locally Lipschitz and it holds

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_t\varphi(x) + \frac{\big(\mathrm{lip} Q_t\varphi(x)\big)^2}{2} \leq 0,$$

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= holds if the space is geodesic

Estimating $W_2(\mu_t, \mu_{t+s})$ by duality

$$\frac{1}{2} W_2^2(\mu_t, \mu_{t+s}) = \sup_{\varphi \in \mathrm{LIP}} \int \varphi f_t \, \mathrm{d}\mathfrak{m} + \int \varphi^c f_{t+s} \, \mathrm{d}\mathfrak{m}$$

$$= \sup_{\psi \in \mathrm{LIP}} \int Q_1 \psi f_{t+s} \, \mathrm{d}\mathfrak{m} - \int \psi f_t \, \mathrm{d}\mathfrak{m}$$

Key computation

$r \mapsto Q_r\psi$ and $r \mapsto f_{t+rs}$ are Lipschitz curves with values in $L^2(X, \mathfrak{m})$.

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$r \mapsto Q_r\psi$ and $r \mapsto f_{t+rs}$ are Lipschitz curves with values in $L^2(X, \mathfrak{m})$. Thus

$$\int Q_1\psi f_{t+s} - \psi f_t \,\mathrm{d}\mathfrak{m}$$

$$= \iint_0^1 \frac{\mathrm{d}}{\mathrm{d}r}\Big(Q_r\psi f_{t+rs}\Big)\,\mathrm{d}r\,\mathrm{d}\mathfrak{m}$$

$$= \iint_0^1 -\frac{(\mathrm{lip}\, Q_r\psi)^2}{2} f_{t+rs} + sQ_r\psi\Delta f_{t+rs}\,\mathrm{d}r\,\mathrm{d}\mathfrak{m}$$

$$\le \iint_0^1 -\frac{(\mathrm{lip}\, Q_r\psi)^2}{2} f_{t+rs} + |DQ_r\psi|\frac{s|Df_{t+rs}|}{f_{t+rs}} f_{t+rs}\,\mathrm{d}r\,\mathrm{d}\mathfrak{m}$$

$$\le \iint_0^1 -\frac{(\mathrm{lip}\, Q_r\psi)^2}{2} f_{t+rs} + \frac{|DQ_r\psi|^2}{2} f_{t+rs} + \frac{s^2}{2}\frac{|Df_{t+rs}|^2}{f_{t+rs}}\,\mathrm{d}r\,\mathrm{d}\mathfrak{m}$$

$$\le \frac{s^2}{2}\int\int_0^1 \frac{|Df_{t+rs}|^2}{f_{t+rs}}\,\mathrm{d}r\,\mathrm{d}\mathfrak{m}.$$

Conclusion of the argument

$$
\begin{aligned}
W_2^2(\mu_t, \mu_{t+s}) &\leq s^2 \int_0^1 \int \frac{|Df_{t+rs}^2|}{f_{t+rs}} \, \mathrm{d}\mathfrak{m} \, \mathrm{d}r \\
&\leq \frac{s^2}{c} \int_0^1 \int |Df_{t+rs}|^2 \, \mathrm{d}\mathfrak{m} \, \mathrm{d}r \\
&\leq \frac{s^2}{c} \int |Df_t|^2 \, \mathrm{d}\mathfrak{m}
\end{aligned}
$$

Spaces with Riemannian Ricci curvature bounded from below

$$
\begin{aligned}
RCD(K,N) &:= CD(K,N) + \text{linearity of the heat flow} \\
&= CD(K,N) + W^{1,2} \text{ is Hilbert}
\end{aligned}
$$

Thank you