Robust utility optimization without compactness

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Ouline

1. Introduction
   - Utility maximization in continuous time financial market
   - Duality methods: standard and robust settings
   - Robust problem under “model compactness”
   - Open questions and motivation
2. Robust problem without model compactness: the complete case
   - An Orlicz-Musielak space formulation
   - Our results
   - Least favorable measure for “linear uncertainty set” by solving a minimization of entropy problem
   - Example
3. Conclusions, open problems

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Continuous Time Financial Market

- Filtered probability space $(\Omega, \mathbb{F}, (\mathcal{F})_{t\leq T}, \mathbb{P})$ ($\mathbb{P}$ reference law).
- Market consists of d stocks and a riskless bond, $S = \left(S^i\right)_{0\leq i\leq d}$.
- S continuous semimartingale.
- A portfolio π is a pair (X_0, H) with H strategy.
- The value of π at time t is $X_t = X_0 + \int_0^t H_u dS_u$.
- $\mathcal{M}^e(S) = \left\{\tilde{\mathbb{P}} \sim \mathbb{P} : S \text{ is a } \tilde{\mathbb{P}}\text{-loc. martingale}\right\} \neq \emptyset$.
  Market is **complete** if $\mathcal{M}^e(S) = \{\mathbb{P}^*\}$.

Admissible wealths starting from x

$$\mathcal{X}(x) = \left\{X \geq 0 : X_t = X_0 + \int_0^t H_u dS_u \text{ with } X_0 \leq x\right\}$$

Utility Function

Utility Functions on $(0, \infty)$

$U : (0, \infty) \to (-\infty, \infty)$ is a *utility function on* $(0, \infty)$, if strictly increasing, strictly concave and continuously differentiable.
In all the sequel, it satisfies **INADA**: $U'(0+) = \infty$ and $U'(\infty) = 0$.

It's **asymptotic elasticity** (KramkovSchachermayer'99) is

$$AE(U) := \limsup_{x \to \infty} \frac{xU'(x)}{U(x)}.$$

Utility maximization problems

Standard utility maximization

Agent tries to maximize expected final utility starting from $x > 0$, under the fixed (subjective) model $\mathbb{Q} = \mathbb{P}$. Value function is

$$u_{\mathbb{Q}}(x) := \sup_{X \in \mathcal{X}(x)} \mathbb{E}^{\mathbb{Q}}[U(X_T)].$$

Robust utility maximization

Actual probabilistic model (law) possibly unknown (model uncertainty) but there is a set $\mathcal{Q}$ of reasonable possible models.
Pessimistic agent tries to maximize expected final utility of the worst-case model. Value function is

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Duality in financial markets

$$V(y) := \sup_{x>0}[U(x) - xy], y > 0 \text{ conjugate of } U.$$

"Supermartingale densities " w.r.t. (subjective) model $\mathbb{Q}$

$\mathcal{Y}_{\mathbb{Q}}(y) := \{Y \geq 0, YX \text{ is a } \mathbb{Q} - \text{supermartingale } \forall X \in \mathcal{X}(1), Y_0 = y\}$.
Generalizes set of densities with respect to $\mathbb{Q}$ of eq. risk measures.

For all $x > 0, X \in \mathcal{X}(x), \mathbb{Q}$,

$$\mathbb{E}^{\mathbb{Q}}[U(X_T)] \leq \inf_{y>0}\left(\inf_{Y\in\mathcal{Y}_{\mathbb{Q}}(y)} \mathbb{E}^{\mathbb{Q}}[V(Y_T)] + xy\right)$$

$$\Longrightarrow v_{\mathbb{Q}}(y) := \inf_{Y\in\mathcal{Y}_{\mathbb{Q}}(y)} \mathbb{E}^{\mathbb{Q}}[V(Y_T)] \text{ candidate conjugate of } u_{\mathbb{Q}}(x),$$

$$v(y) := \inf_{\mathbb{Q}\in\mathcal{Q}} v_{\mathbb{Q}}(y) \text{ candidate conjugate of } u(x).$$

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Known results: non-robust case

- [KaratzasLehoczkyShreve](87) (complete market) , [KramkovSchachermayer](99,03) (incomplete market)
- Under (finiteness assumptions), duality holds:

$$
\begin{aligned}
v_{\mathbb{Q}}(y) &= \sup_{x\geq 0}[u_{\mathbb{Q}}(x) - xy] \,,\, y > 0 \\
u_{\mathbb{Q}}(x) &= \inf_{y\geq 0}[v_{\mathbb{Q}}(y) + xy] \,,\, x > 0
\end{aligned}
$$

  and $u_{\mathbb{Q}}, v_{\mathbb{Q}}$ have nice properties.
- $v_{\mathbb{Q}}(y) := \inf_{Y \in \mathcal{Y}_{\mathbb{Q}}(y)} \mathbb{E}^{\mathbb{Q}}[V(Y_T)]$ attained.
- Under additional assumptions (e.g. $AE(U) < 1$), for each x,

$$
\hat{X}_T(x) = (U')^{-1}\left(\hat{Y}_T(\hat{y})\right), \text{ where } \hat{y} \in \partial u_{\mathbb{Q}}(x)
$$

  and $\hat{Y}(\hat{y})$ the above minimum. Also, $\hat{X}(x)\hat{Y}(\hat{y})$ is a u.i. $\mathbb{Q}$-martingale.

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Robust case under model compactness assumption

- Dual involves $v(y) = \inf_{\mathbb{Q}\in\mathcal{Q}} \inf_{Y\in\mathcal{Y}_{\mathbb{P}}(y)} \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} V\left(\frac{Y_T}{d\mathbb{Q}/d\mathbb{P}}\right)\right]$.
- Primal requires Minmax: $\sup_{X\in\mathcal{X}(x)} \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}\left[U(X_T)\right] = \inf_{\mathbb{Q}\in\mathcal{Q}} u_{\mathbb{Q}}(x)$.

Conditions on $\mathcal{Q}$ are needed. Shied& Wu 05 consider:

1. $\mathcal{Q}$ convex,
2. $\mathbb{P}(A) = 0 \iff \mathbb{Q}(A) = 0 \, \forall \mathbb{Q} \in \mathcal{Q}$, and
3. $\frac{d\mathcal{Q}}{d\mathbb{P}} := \left\{\frac{d\mathbb{Q}}{d\mathbb{P}} : \mathbb{Q} \in \mathcal{Q}\right\}$ closed in $L^0(\mathbb{P})$ (equiv. $\sigma(L^1, L^\infty)-$**compact**).

Theorem ([SchiedWu05] (see also Gundel 03))

Then, minmax equality holds and u, v are conjugate. Under additional assumptions (e.g. $AE(U) < 1$), everything is attained:

$$u(x) = u_{\hat{\mathbb{Q}}}(x), \quad \hat{X}_T = (U')^{-1}(\hat{Y}_T/\hat{Z}_T)$$

where $\hat{y} \in \partial u(x)$, $\hat{Y} \in \mathcal{Y}(\hat{y})$ and the pair $\left(\hat{Z} = \frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}, \hat{Y}\right)$ attains the double infimum in the dual problem for such $(x, \hat{y})$.

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Open questions and our motivation

- No general characterization of $\hat{\mathbb{Q}}$ (except when independent of U).
- There are simple (and reasonable) uncertainty sets, that are not **weakly compact** in $L^1(\mathbb{P})$. e.g.:

$$\mathcal{Q} = \{\mathbb{Q} \ll \mathbb{P} : \mathbb{E}^{\mathbb{Q}}[S_T] \geq A\}, \quad A > 0.$$

  More generally, $\mathcal{Q}$ determined by "moment" or distributional constraints

$$\mathcal{Q} = \bigcap_i \{\mathbb{Q} \ll \mathbb{P} : \mathbb{E}^{\mathbb{Q}}[F_i(S)] \in C_i\}$$

  arise naturally and may fail to be compact.
- **Goal:** Find a framework to study the above problems.
- **Goal:** use **techniques of constrained entropy minimization to describe least favorable measure**.

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Orlicz-Musielak space associated with the robust problem

.

In the sequel, we assume U satisfies **INADA**, $U \geq 0$ and $U(\infty) = \infty$.
Moreover, **market is complete:** $\mathcal{M}^e = \{\mathbb{P}^*\}$,

We have

$$u(x) = \sup_{X \in \mathcal{X}(x)} \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}} \left[U \left(X_T \right) \right] \leq \inf_{y \geq 0} \left[\inf_{\mathbb{Q} \in \mathcal{Q}_e} \mathbb{E}^{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} V \left(\frac{yY_T}{\frac{d\mathbb{Q}}{d\mathbb{P}}} \right) \right] + xy \right]$$

where $Y := \frac{d\mathbb{P}^*}{d\mathbb{P}}$.

Thus, we only are only concerned with $\mathbb{Q} \in \mathcal{Q}$ such that $Z := \frac{d\mathbb{Q}}{d\mathbb{P}}$ belongs to the **Orlicz-Musielak** space:

$$L_\eta = \left\{ Z \in L^0 \text{ s.t. } \exists \alpha > 0, \mathbb{E}^{\mathbb{P}} \left[\eta \left(\alpha Z, \cdot \right) \right] < \infty \right\}$$

where $\eta(z, \omega) := |z| V \left(\frac{Y_T(\omega)}{|z|} \right)$.

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Some Properties

- $\mathbb{P}(d\omega)$ a.s., $z \mapsto \eta(z,\omega) := |z| V\left(\frac{Y_T(\omega)}{|z|}\right)$ is a Young function (convex, l.s.c., even, ≥ 0 and null (only) at 0), $\eta(z,\omega)/|z| \to \infty$ as $|z| \to \infty$.
- $\mathbb{P}(d\omega)$ a.s. conjugate of $\eta(z,\omega)$ is $z \mapsto \eta^*(z,\omega) := Y_T(\omega)U^{-1}(|z|)$
- $L_\eta = \left\{Z \in L^0 \text{ s.t. } \exists \alpha > 0, \mathbb{E}^{\mathbb{P}}\left[\eta\left(\alpha Z, \cdot\right)\right] < \infty\right\}$ is Banach with norm

$$\|Z\|_\eta = \text{ínf}\left\{\beta > 0 : \mathbb{E}^{\mathbb{P}}\left[\eta\left(Z/\beta, \cdot\right)\right] \leq 1\right\},$$

  and is continuously embedded in L^1.
- Under $\triangle_2$ condition on η plus other conditions (including that η^* is $\triangle_2$ and holding in particular if $AE(U) < 1$), η, η^* are $N-$functions,

$$[L_\eta]^* \text{ is isomorphic to } [L_{\eta^*}] \text{ and } [L_{\eta^*}]^* \text{ is isomorphic to } [L_\eta].$$

  In particular, L_η is **reflexive.**
- Bounded sets of L_η are u.i.

([Kozek] 76,80 and [Musielak] 83 for background on O.M. spaces))

Results

Theorem (Backhoff, F', 2012)

Assume

- $\mathcal{Q}$ *is countably convex*
- $\mathbb{P}(A) = 0 \iff \forall \mathbb{Q} \in \mathcal{Q} \quad \mathbb{Q}(A) = 0$
- $\frac{d\mathcal{Q}}{d\mathbb{P}} \cap L_\eta(\mathbb{P})$ *is* $\sigma(L_\eta, L_{\eta^*})-$*closed.*

Moreover, assume that U *satisfies* **INADA**, $U \geq 0$ *and* $U(\infty) = \infty$, *that* η *and* η^* *satisfy* $\triangle_2$ *(so that* L_η *is reflexive.)*
Then, for each x *with* $u(x) < \infty$

$$Z = \frac{d\mathbb{Q}}{d\mathbb{P}} \in \frac{d\mathcal{Q}}{d\mathbb{P}} \cap L_\eta \mapsto u_{\mathbb{Q}}(x) \geq C_x \|Z\|_\eta .$$

We deduce that minmax equality holds, and $u, -v$ *are conjugate.*
Moreover,

$$u(x) = u_{\hat{\mathbb{Q}}}(x) , \quad \hat{X}_T = (U')^{-1}(Y_T/\hat{Z}_T)$$

where $\hat{y} \in \partial u(x)$ *and* $\hat{Z} = \frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}$ *attains the infimum in the dual problem*

Convex integral functionals associated with the dual problem

By previous Theorem, for each x s.t. $u(x) < \infty$, solution $(\hat{y}, \hat{Q})$ of dual problem satisfies

$$u(x) = x\hat{y} + \Phi_{\hat{y}}(\hat{\mathbb{Q}}) = x\hat{y} + \inf_{\mathbb{Q}\in\mathcal{Q}} \Phi_{\hat{y}}(\mathbb{Q})$$

where

$$\Phi_y(\mathbb{Q}) := \int_\Omega \eta_y \left(\frac{d\mathbb{Q}}{d\mathbb{P}}(,\omega), \omega \right) \mathbb{P}(d\omega)$$

and $\eta_y(z,\omega) := |z| V\left(\frac{yY_T(\omega)}{|z|}\right)$.

Problem enters the framework of **minimization of convex integral functionals** (entropies):

Rockafellar $\sim$70, Csiszar, Föllmer $\sim$ 80's, BorweinLewis $\sim$ 90's, ...Léonard, $\sim$ 07.

Uncertainty set as linear/convex constraints

We consider uncertainty set $\mathcal{Q}$ such that

$$\frac{d\mathcal{Q}}{d\mathbb{P}} \cap L_\eta = \left\{ \mathbb{Q} \ll \mathbb{P} : \Theta\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right) \in C \right\}$$

for

$$\Theta : L_\eta(\Omega, \mathbb{P}) \to E$$

a linear operator of integral type, taking values in some vectorial space E (possibly ∞-dim.) and $C \subseteq E$ a convex subset.

More precisely, there is a measurable function $\theta : \Omega \to E$ such that

$$\Theta(Z) = \mathbb{E}^{\mathbb{P}}(Z\theta) \in E$$

(makes sense in E by Hölder's inequality if $\langle g, \theta \rangle_{E',E} \in L_{\eta^*}$ for each $g \in E'$ (alg. dual)).

This includes moment constraints on " observables" of any dimension; in particular, any restriction (or belief) of distributional type on prices or assets can be described in this way

Uncertainty set as linear/convex constraints

Minimization problem is embedded into the space $\mathcal{M}_f$ of finite signed measures $\mathbb{M}$ on Ω:

$$\Phi_y(\mathbb{M}) := \begin{cases} \int_\Omega \eta_y\left(\frac{d\mathbb{M}}{d\mathbb{P}}(,\omega),\omega\right)\mathbb{M}(d\omega) & \text{if } \mathbb{M} \geq 0 \text{ and } \mathbb{M} \ll \mathbb{P} \\ +\infty & \text{otherwise} \end{cases},$$

adding the supplementary constraint $\mathbb{E}^{\mathbb{P}}(\frac{d\mathbb{M}}{d\mathbb{P}}) = 1$

Summarizing, we want the solution of

PC

$$\textit{Minimize } \Phi_{\hat{y}}(\mathbb{M}) \quad \text{subject to} \quad \Theta_1(\mathbb{M}) \in C_1 \quad , \mathbb{M} \in \mathcal{M}_f$$

where $\Theta_1(\mathbb{M}) = (\int_\Omega \theta d\mathbb{M}, \int_\Omega 1 d\mathbb{M}) \in E_1 = E \times \mathbb{R}$ and $C_1 = C \times \{1\}$

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Dual of the Entropy minimization Problem

Dual Problem
(i.e. a “Dual of the Dual” of our original utility maximization problem):

DC

$$\sup\left\{\inf_{x\in\bar{E}_1\cap C_1}\langle g,x\rangle-\int\eta^*(\langle g,\theta(\cdot)\rangle)d\mathbb{P}:g\in E_1^*\right\}$$

where E_1^* is completion of E_1' w.r.t. $\|\langle g,\theta\rangle_{E_1',E_1}\|_{L_{\eta^*}}$ (norm under reasonable assumptions), and $\bar{E}_1$ topological dual.

Case of moment constraints, dimension $=$ number of “real observables ” $+1$, finite in many interesting problems.

Finding the minimizer

We use

Theorem ([Léonard08])

Under above assumptions:

- *There is dual equality* $PC = DC$
- *If* $C_1 \cap \Theta_1(dom(\Phi_{\hat{y}})) \neq \emptyset$, *PC has a unique solution in* $L_\eta(\mathbb{P})$
- *If moreover* $C_1 \cap icor(\Theta_1(dom(\Phi_{\hat{y}}))) \neq \emptyset$ *the solution of* ***PC*** *is given by*

$$\hat{\mathbb{Q}} = \frac{d\eta^*}{dz}(< \tilde{g}, \theta >)d\mathbb{P}.$$

where $\tilde{g}$ *solves the (extended) dual.*

Here, $icor(A) = \{a \in A | \forall x \in aff(A), \exists t > 0 \text{ tq. } a + t(x - a) \in A\}$.

1. Introduction
   - Utility maximization in continuous time financial market
   - Duality methods: standard and robust settings
   - Robust problem under "model compactness"
   - Open questions and motivation

2. Robust problem without model compactness: the complete case
   - An Orlicz-Musielak space formulation
   - Our results
   - Least favorable measure for "linear uncertainty set" by solving a minimization of entropy problem
   - Example

3. Conclusions, open problems

Example

Consider on $\left(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t=0}^{T}, \mathbb{P}\right)$, and for $t \leq T$, the diffusion

$$\begin{aligned} dS_t &= S_t\{bdt + \sigma dW_t\} \\ S_0 &= 1 \end{aligned}$$

(most elementary Black-Scholes).

This model is complete, with risk neutral measure $d\mathbb{P}^* = \exp\left\{-\frac{b}{\sigma} W_T - \frac{b^2}{2\sigma^2} T\right\} d\mathbb{P}$. Under $\mathbb{P}^*$, $S_T \sim lognormal(-\frac{\sigma^2}{2} T, \sigma^2 T)$.

Example

- We take $U(x) = 2\sqrt{x}$, $x \in (0, \infty)$, thus $L_\eta = L^2$.
- For $A \geq 0$, consider the uncertainty set

$$\mathcal{Q} = \{\mathbb{Q} \ll \mathbb{P} : \mathbb{E}^{\mathbb{Q}}(S_T) \geq A\}$$

  which is not closed in L^0 and not bounded in L^2, but weakly closed in L^2.
- Constraint qualification condition is met if $e^{\sigma^2 T} > A > 1$

Solution

From the dual representation of the corresponding Φ, it follows:

$$\inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} V\left(yY_T\frac{d\mathbb{P}}{d\mathbb{Q}}\right)\right] = \sup_{\mathbb{R}^2}\left[z_1 + Az_2 - \frac{y}{4}\mathbb{E}^{\mathbb{P}}\left((z_1+S_T z_2)^2 \mathbb{1}_{z_1+S_T z_2>0}\right)\right]$$

Right-hand side can be solved, and by means if the duality relation between u and v, we get:

$$u(x) = 2\sqrt{x\left(1+\frac{(A-1)^2}{e^{\sigma^2 T}-1}\right)},$$

$$\hat{\mathbb{Q}}(d\omega) = \frac{e^{\sigma^2 T} - A + S_T(A-1)}{e^{\sigma^2 T}-1}\mathbb{P}(d\omega)$$

and

$$\hat{X}_T := x\frac{\left(e^{\sigma^2 T} - A + S_T(A-1)\right)^2}{\left(e^{\sigma^2 T}-1+(A-1)^2\right)\left(e^{\sigma^2 T}-1\right)}.$$

- Functional setting and methodology to solve robust problem in complete market beyond model compactness framework is proposed.
- Least favorable measure can be explicitly (or numerically) computed when uncertainty is determined by finitely many moment constraints.

- Approach not readily extendable to utility functions not bounded from below or defined in $(-\infty, \infty)$
- Incomplete case : Dual problem can be embedded in a convex modular space with explicit topology (Banach lattice) and dual space. Coercivity holds, but space is not reflexive.

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