Robust utility optimization without compactness

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MAY, 2013
Outline

1. Introduction
   - Utility maximization in continuous time financial market
   - Duality methods: standard and robust settings
   - Robust problem under “model compactness”
   - Open questions and motivation

2. Robust problem without model compactness: the complete case
   - An Orlicz-Musielak space formulation
   - Our results
   - Least favorable measure for “linear uncertainty set” by solving a minimization of entropy problem
   - Example

3. Conclusions, open problems
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Continuous Time Financial Market

- Filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \leq T}, \mathbb{P})\) (\(\mathbb{P}\) reference law).
- Market consists of \(d\) stocks and a riskless bond, \(S = (S^i)_{0 \leq i \leq d}\).
- \(S\) continuous semimartingale.
- A portfolio \(\pi\) is a pair \((X_0, H)\) with \(H\) strategy.
- The value of \(\pi\) at time \(t\) is \(X_t = X_0 + \int_0^t H_u dS_u\).
- \(\mathcal{M}^e(S) = \{\tilde{\mathbb{P}} \sim \mathbb{P} : S\) is a \(\tilde{\mathbb{P}}\)-loc. martingale\} \(\neq \emptyset\).
  Market is **complete** if \(\mathcal{M}^e(S) = \{\mathbb{P}^*\}\).

**Admissible wealths starting from \(x\)**

\[
\mathcal{X}(x) = \left\{ X \geq 0 : X_t = X_0 + \int_0^t H_u dS_u \text{ with } X_0 \leq x \right\}
\]
Utility Functions on \((0, \infty)\)

\(U : (0, \infty) \to (-\infty, \infty)\) is a utility function on \((0, \infty)\), if strictly increasing, strictly concave and continuously differentiable. In all the sequel, it satisfies INADA: \(U'(0+) = \infty\) and \(U'(\infty) = 0\).

It’s asymptotic elasticity (KramkovSchachermayer’99) is

\[
AE(U) := \limsup_{x \to \infty} \frac{xU'(x)}{U(x)}.
\]
Utility maximization problems

Standard utility maximization

Agent tries to maximize expected final utility starting from $x > 0$, under the fixed (subjective) model $\mathbb{Q} = \mathbb{P}$. Value function is

$$u_{\mathbb{Q}}(x) := \sup_{X \in \mathcal{X}(x)} \mathbb{E}^\mathbb{Q}[U(X_T)].$$

Robust utility maximization

Actual probabilistic model (law) possibly unknown (model uncertainty) but there is a set $\mathbb{Q}$ of reasonable possible models. Pessimistic agent tries to maximize expected final utility of the worst-case model. Value function is

$$u(x) := \sup_{X \in \mathcal{X}(x)} \inf_{\mathbb{Q} \in \mathbb{Q}} \mathbb{E}^\mathbb{Q}[U(X_T)].$$
Utility maximization problems

Standard utility maximization
Agent tries to maximize expected final utility starting from $x > 0$, under the fixed (subjective) model $Q = P$. Value function is

$$u_Q(x) := \sup_{X \in \mathcal{X}(x)} \mathbb{E}^Q[U(X_T)].$$

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Duality in financial markets

\[ V(y) := \sup_{x>0} [U(x) - xy], \ y > 0 \text{ conjugate of } U. \]

“Supermartingale densities” w.r.t. (subjective) model \( Q \)

\[ \mathcal{Y}_Q(y) := \{ Y \geq 0, YX \text{ is a } Q - \text{supermartingale } \forall X \in \mathcal{X}(1), Y_0 = y \}. \]

Generalizes set of densities with respect to \( Q \) of eq. risk measures.

For all \( x > 0, X \in \mathcal{X}(x), Q, \)

\[ \mathbb{E}^Q[U(X_T)] \leq \inf_{y>0} \left( \inf_{Y \in \mathcal{Y}_Q(y)} \mathbb{E}^Q[V(Y_T)] + xy \right) \]

\[ \implies v_Q(y) := \inf_{Y \in \mathcal{Y}_Q(y)} \mathbb{E}^Q[V(Y_T)] \text{ candidate conjugate of } u_Q(x), \]

\[ v(y) := \inf_{Q \in \mathcal{Q}} v_Q(y) \text{ candidate conjugate of } u(x). \]
Duality in financial markets

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“Supermartingale densities ” w.r.t. (subjective) model \( \mathcal{Q} \)

\[ \mathcal{Y}_\mathcal{Q}(y) := \{ Y \geq 0, \ YX \text{ is a } \mathcal{Q} - \text{supermartingale } \forall X \in \mathcal{X}(1), \ Y_0 = y \}. \]

Generalizes set of densities with respect to \( \mathcal{Q} \) of eq. risk measures.

For all \( x > 0, \ X \in \mathcal{X}(x), \mathcal{Q}, \)

\[ \mathbb{E}^\mathcal{Q}[U(X_T)] \leq \inf_{y>0} \left( \inf_{Y \in \mathcal{Y}_\mathcal{Q}(y)} \mathbb{E}^\mathcal{Q}[V(Y_T)] + xy \right) \]

\[ \implies v_\mathcal{Q}(y) := \inf_{Y \in \mathcal{Y}_\mathcal{Q}(y)} \mathbb{E}^\mathcal{Q}[V(Y_T)] \text{ candidate conjugate of } u_\mathcal{Q}(x), \]

\[ v(y) := \inf_{\mathcal{Q} \in \mathcal{Q}} v_\mathcal{Q}(y) \text{ candidate conjugate of } u(x). \]
Duality in financial markets

\[ V(y) := \sup_{x > 0} [U(x) - xy], \quad y > 0 \text{ conjugate of } U. \]

“Supermartingale densities ” w.r.t. (subjective) model \( \mathbb{Q} \)

\[ \mathcal{Y}_\mathbb{Q}(y) := \{ Y \geq 0, \ YX \text{ is a } \mathbb{Q} - \text{supermartingale} \ \forall X \in \mathcal{X}(1), \ Y_0 = y \}. \]

Generalizes set of densities with respect to \( \mathbb{Q} \) of eq. risk measures.

For all \( x > 0, \ X \in \mathcal{X}(x), \ \mathbb{Q}, \)

\[
\mathbb{E}^\mathbb{Q}[U(X_T)] \leq \inf_{y > 0} \left( \inf_{Y \in \mathcal{Y}_\mathbb{Q}(y)} \mathbb{E}^\mathbb{Q}[V(Y_T)] + xy \right)
\]

\[ \implies v_\mathbb{Q}(y) := \inf_{Y \in \mathcal{Y}_\mathbb{Q}(y)} \mathbb{E}^\mathbb{Q}[V(Y_T)] \text{ candidate conjugate of } u_\mathbb{Q}(x), \]

\[ v(y) := \inf_{\mathbb{Q} \in \mathbb{Q}} v_\mathbb{Q}(y) \text{ candidate conjugate of } u(x). \]
Known results: non-robust case

- [KaratzasLehoczkyShreve](87) (complete market),
- [KramkovSchachermayer](99,03) (incomplete market)

Under (finiteness assumptions), duality holds:

\[ v_Q(y) = \sup_{x \geq 0} [u_Q(x) - xy] , \ y > 0 \]
\[ u_Q(x) = \inf_{y \geq 0} [v_Q(y) + xy] , \ x > 0 \]

and \( u_Q, v_Q \) have nice properties.

- \( v_Q(y) := \inf_{Y \in \mathcal{Y}_Q(y)} \mathbb{E}^Q[V(Y_T)] \) attained.
- Under additional assumptions (e.g. \( \text{AE}(U) < 1 \)), for each \( x \),

\[ \hat{X}_T(x) = (U')^{-1} \left( \hat{Y}_T(\hat{y}) \right) , \text{ where } \hat{y} \in \partial u_Q(x) \]

and \( \hat{Y}(\hat{y}) \) the above minimum. Also, \( \hat{X}(x) \hat{Y}(\hat{y}) \) is a u.i. \( \mathbb{Q} \)-martingale.
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Robust case under model compactness assumption

- Dual involves
  \[ v(y) = \inf_{Q \in \mathcal{Q}} \inf_{Y \in \mathcal{Y}_P(y)} \mathbb{E} \left[ \frac{dQ}{dP} \, V \left( \frac{Y_T}{dQ/dP} \right) \right]. \]

- Primal requires Minmax:
  \[ \sup_{X \in \mathcal{X}(x)} \inf_{Q \in \mathcal{Q}} \mathbb{E}^Q \left[ U(X_T) \right] = \inf_{Q \in \mathcal{Q}} u_Q(x). \]

Conditions on \( \mathcal{Q} \) are needed. Shied & Wu 05 consider:

1. \( \mathcal{Q} \) convex,
2. \( P(A) = 0 \iff Q(A) = 0 \forall Q \in \mathcal{Q}, \) and
3. \( \frac{dQ}{dP} := \left\{ \frac{dQ}{dP} : Q \in \mathcal{Q} \right\} \) closed in \( L^0(P) \) (equiv. \( \sigma(L^1, L^\infty) - \text{compact} \)).

Theorem ([SchiedWu05] (see also Gundel 03))

Then, minmax equality holds and \( u, v \) are conjugate. Under additional assumptions (e.g. \( AE(U) < 1 \)), everything is attained:

\[ u(x) = u_{\hat{Q}}(x), \quad \hat{X}_T = (U')^{-1}(\hat{Y}_T/\hat{Z}_T) \]

where \( \hat{y} \in \partial u(x), \hat{Y} \in \mathcal{Y}(\hat{y}) \) and the pair \( (\hat{Z} = \frac{d\hat{Q}}{dP}, \hat{Y}) \) attains the double infimum in the dual problem for such \( (x, \hat{y}) \).
Robust case under model compactness assumption

- Dual involves \( v(y) = \inf_{Q \in \mathcal{Q}} \inf_{Y \in \mathcal{Y}_P(y)} \mathbb{E} \left[ \frac{dQ}{dP} V \left( \frac{Y_T}{\frac{dQ}{dP}} \right) \right] \).
- Primal requires Minmax:
  \[
  \sup_{X \in \mathcal{X}(x)} \inf_{Q \in \mathcal{Q}} \mathbb{E}^Q [U(X_T)] = \inf_{Q \in \mathcal{Q}} u_Q(x).
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Then, minmax equality holds and \( u, v \) are conjugate. Under additional assumptions (e.g. \( AE(U) < 1 \)), everything is attained:

\[
  u(x) = u_\hat{Q}(x), \quad \hat{X}_T = (U')^{-1}(\hat{Y}_T/\hat{Z}_T)
\]

where \( \hat{y} \in \partial u(x) \), \( \hat{Y} \in \mathcal{Y}(\hat{y}) \) and the pair \( \left( \hat{Z} = \frac{d\hat{Q}}{dP}, \hat{Y} \right) \) attains the double infimum in the dual problem for such \((x, \hat{y})\).
Robust case under model compactness assumption

- Dual involves $v(y) = \inf_{Q \in Q} \inf_{Y \in Y_y} \mathbb{E} \left[ \frac{dQ}{dP} V \left( \frac{Y_T}{dQ/dP} \right) \right]$.
- Primal requires Minmax: $\sup_{X \in X(x)} \inf_{Q \in Q} \mathbb{E}^Q [U(X_T)] = \inf_{Q \in Q} u_Q(x)$.

Conditions on $Q$ are needed. Shied & Wu 05 consider:

1. $Q$ convex,
2. $P(A) = 0 \iff Q(A) = 0 \ \forall Q \in Q$, and
3. $\frac{dQ}{dP} := \left\{ \frac{dQ}{dP} : Q \in Q \right\}$ closed in $L^0(P)$ (equiv. $\sigma(L^1, L^\infty)$-compact).

Theorem ([SchiedWu05] (see also Gundel 03))

Then, minmax equality holds and $u, v$ are conjugate. Under additional assumptions (e.g. $AE(U) < 1$), everything is attained:

$$u(x) = u_{\hat{Q}}(x), \quad \hat{X}_T = (U')^{-1}(\hat{Y}_T/\hat{Z}_T)$$

where $\hat{y} \in \partial u(x)$, $\hat{Y} \in Y(\hat{y})$ and the pair $\left( \hat{Z} = \frac{d\hat{Q}}{dP}, \hat{Y} \right)$ attains the double infimum in the dual problem for such $(x, \hat{y})$. 
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Open questions and our motivation

- No general characterization of $\hat{Q}$ (except when independent of $U$).
- There are simple (and reasonable) uncertainty sets, that are not weakly compact in $L^1(\mathbb{P})$. e.g.:

\[ Q = \{ Q \ll P : \mathbb{E}^Q[S_T] \geq A \}, \quad A > 0. \]

More generally, $Q$ determined by “moment” or distributional constraints

\[ Q = \bigcap_i \{ Q \ll P : \mathbb{E}^Q[F_i(S)] \in C_i \} \]

arise naturally and may fail to be compact.

**Goal:** Find a framework to study the above problems.

**Goal:** use techniques of constrained entropy minimization to describe least favorable measure.
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In the sequel, we assume $U$ satisfies **INADA**, $U \geq 0$ and $U(\infty) = \infty$. Moreover, **market is complete**: $\mathcal{M}^e = \{\mathbb{P}^*\}$.

We have

$$u(x) = \sup_{X \in \mathcal{X}(x)} \inf_{Q \in \mathcal{Q}} \mathbb{E}^Q [U(X_T)] \leq \inf_{y \geq 0} \left[ \inf_{Q \in \mathcal{Q}_e} \mathbb{E}^P \left[ \frac{dQ}{dP} \left( \frac{yY_T}{\frac{dQ}{dP}} \right) \right] + xy \right]$$

where $Y := \frac{dP^*}{dP}$.

Thus, we only are only concerned with $Q \in \mathcal{Q}$ such that $Z := \frac{dQ}{dP}$ belongs to the **Orlicz-Musielak** space:

$$L_\eta = \left\{ Z \in L^0 \text{ s.t. } \exists \alpha > 0, \mathbb{E}^P \left[ \eta(\alpha Z, \cdot) \right] < \infty \right\}$$

where $\eta(z, \omega) := |z| V \left( \frac{Y_T(\omega)}{|z|} \right)$. 

Orlicz-Musielak space associated with the robust problem.

In the sequel, we assume \( U \) satisfies \( \text{INADA}, \ U \geq 0 \) and \( U(\infty) = \infty \). Moreover, \textit{market is complete}: \( \mathcal{M}^e = \{\mathbb{P}^*\} \),

We have

\[
 u(x) = \sup_{X \in \mathcal{X}(x)} \inf_{Q \in \mathcal{Q}} \mathbb{E}^Q [U(X_T)] \leq \inf_{y \geq 0} \left[ \inf_{Q \in \mathcal{Q}_e} \mathbb{E}^P \left[ \frac{dQ}{dP} \nu \left( \frac{yY_T}{dQ/dP} \right) \right] + xy \right]
\]

where \( Y := \frac{dP^*}{dP} \).

Thus, we only are only concerned with \( Q \in \mathcal{Q} \) such that \( Z := \frac{dQ}{dP} \) belongs to the \textit{Orlicz-Musielak} space:

\[
 L_\eta = \left\{ Z \in L^0 \text{ s.t. } \exists \alpha > 0, \mathbb{E}^P [\eta(\alpha Z, \cdot)] < \infty \right\}
\]

where \( \eta(z, \omega) := |z| V \left( \frac{Y_T(\omega)}{|z|} \right) \).
Some Properties

- \( \mathbb{P}(d\omega) \text{ a.s.}, \ z \mapsto \eta(z, \omega) := |z| V \left( \frac{Y_T(\omega)}{|z|} \right) \) is a Young function (convex, l.s.c., even, \( \geq 0 \) and null (only) at 0), \( \eta(z, \omega)/|z| \to \infty \) as \( |z| \to \infty \).

- \( \mathbb{P}(d\omega) \text{ a.s. conjugate of } \eta(z, \omega) \) is \( z \mapsto \eta^*(z, \omega) := Y_T(\omega) U^{-1}(|z|) \).

- \( L_\eta = \{ Z \in L^0 \text{ s.t. } \exists \alpha > 0, \mathbb{E}^{\mathbb{P}}[\eta(\alpha Z, \cdot)] < \infty \} \) is Banach with norm

\[
\|Z\|_\eta = \inf \left\{ \beta > 0 : \mathbb{E}^{\mathbb{P}}[\eta(Z/\beta, \cdot)] \leq 1 \right\},
\]

and is continuously embedded in \( L^1 \).

- Under \( \Delta_2 \) condition on \( \eta \) plus other conditions (including that \( \eta^* \) is \( \Delta_2 \) and holding in particular if \( AE(U) < 1 \)), \( \eta, \eta^* \) are \( N \)–functions,

\[
[L_\eta]^* \text{ is isomorphic to } [L_{\eta^*}] \text{ and } [L_{\eta^*}]^* \text{ is isomorphic to } [L_\eta].
\]

In particular, \( L_\eta \) is reflexive.

- Bounded sets of \( L_\eta \) are u.i.

([Kozek] 76,80 and [Musielak] 83 for background on O.M. spaces))
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Results

Theorem (Backhoff, F’, 2012)

Assume

- $Q$ is countably convex
- $\mathbb{P}(A) = 0 \iff \forall Q \in Q \quad Q(A) = 0$
- $\frac{dQ}{dP} \cap L_\eta(P)$ is $\sigma(L_\eta, L_\eta^*)$-closed.

Moreover, assume that $U$ satisfies INADA, $U \geq 0$ and $U(\infty) = \infty$, that $\eta$ and $\eta^*$ satisfy $\triangle_2$ (so that $L_\eta$ is reflexive.)

Then, for each $x$ with $u(x) < \infty$

$$Z = \frac{dQ}{dP} \in \frac{dQ}{dP} \cap L_\eta \mapsto u_Q(x) \geq C_x \|Z\|_\eta.$$

We deduce that minmax equality holds, and $u, -v$ are conjugate.

Moreover,

$$u(x) = u_\hat{Q}(x), \quad \hat{X}_T = (U')^{-1}(Y_T/\hat{Z}_T)$$

where $\hat{y} \in \partial u(x)$ and $\hat{Z} = \frac{d\hat{Q}}{d\hat{P}}$ attains the infimum in the dual problem.
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Convex integral functionals associated with the dual problem

By previous Theorem, for each \( x \) s.t. \( u(x) < \infty \), solution \((\hat{y}, \hat{Q})\) of dual problem satisfies

\[
u(x) = x\hat{y} + \Phi_{\hat{y}}(\hat{Q}) = x\hat{y} + \inf_{Q \in Q} \Phi_{\hat{y}}(Q)\]

where

\[
\Phi_{\hat{y}}(Q) := \int_{\Omega} \eta_{\hat{y}} \left( \frac{dQ}{dP}(\omega), \omega \right) P(d\omega)
\]

and \( \eta_{\hat{y}}(z, \omega) := |z| V \left( \frac{y^T(\omega)}{|z|} \right) \).

Problem enters the framework of minimization of convex integral functionals (entropies):

Rockafellar \( \sim 70 \), Csiszar, Föllmer \( \sim 80\)’s, BorweinLewis \( \sim 90\)’s, ...

...Léonard, \( \sim 07 \).
Uncertainty set as linear/convex constraints

We consider uncertainty set $Q$ such that

$$\frac{dQ}{dP} \cap L_\eta = \left\{ Q \ll P : \Theta \left( \frac{dQ}{dP} \right) \in C \right\}$$

for

$$\Theta : L_\eta(\Omega, P) \to E$$

a linear operator of integral type, taking values in some vectorial space $E$ (possibly $\infty$-dim.) and $C \subseteq E$ a convex subset.

More precisely, there is a measurable function $\theta : \Omega \to E$ such that

$$\Theta(Z) = \mathbb{E}^P(Z\theta) \in E$$

(makes sense in $E$ by Hölder’s inequality if $\langle g, \theta \rangle_{E', E} \in L_{\eta^*}$ for each $g \in E'$ (alg. dual)).

This includes moment constraints on “observables” of any dimension; in particular, any restriction (or belief) of distributional type on prices or assets can be described in this way.
Uncertainty set as linear/convex constraints

Minimization problem is embedded into the space $\mathcal{M}_f$ of finite signed measures $\mathcal{M}$ on $\Omega$:

$$\Phi_y(\mathcal{M}) := \begin{cases} \int_{\Omega} \eta_y \left( \frac{d\mathcal{M}}{dP}(\omega), \omega \right) \mathcal{M}(d\omega) & \text{if } \mathcal{M} \geq 0 \text{ and } \mathcal{M} \ll P \\ +\infty & \text{otherwise} \end{cases},$$

adding the supplementary constraint $\mathbb{E}^P \left( \frac{d\mathcal{M}}{dP} \right) = 1$

Summarizing, we want the solution of

$$\text{Minimize } \Phi_y(\mathcal{M}) \text{ subject to } \Theta_1(\mathcal{M}) \in C_1, \mathcal{M} \in \mathcal{M}_f$$

where $\Theta_1(\mathcal{M}) = (\int_{\Omega} \theta d\mathcal{M}, \int_{\Omega} 1 d\mathcal{M}) \in E_1 = E \times \mathbb{R}$ and $C_1 = C \times \{1\}$.
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adding the supplementary constraint $E^P(\frac{d\mathcal{M}}{dP}) = 1$

Summarizing, we want the solution of

**PC**

Minimize $\Phi_y(\mathcal{M})$ subject to $\Theta_1(\mathcal{M}) \in C_1$, $\mathcal{M} \in \mathcal{M}_f$

where $\Theta_1(\mathcal{M}) = (\int_\Omega \theta d\mathcal{M}, \int_\Omega 1 d\mathcal{M}) \in E_1 = E \times \mathbb{R}$ and $C_1 = C \times \{1\}$
Dual of the Entropy minimization Problem

Dual Problem
(i.e. a “Dual of the Dual” of our original utility maximization problem):

\[
\sup \left\{ \inf_{x \in \bar{E}_1 \cap C_1} \langle g, x \rangle - \int \eta^* (\langle g, \theta(\cdot) \rangle) d\mathbb{P} : g \in E_1^* \right\}
\]

where \( E_1^* \) is completion of \( E_1' \) w.r.t. \( \| \langle g, \theta \rangle_{E_1', E_1} \|_{L_{\eta^*}} \) (norm under reasonable assumptions), and \( \bar{E}_1 \) topological dual.

Case of moment constraints, dimension = number of “real observables” + 1, finite in many interesting problems.
Finding the minimizer

We use

**Theorem ([Léonard08])**

*Under above assumptions:*

- **There is dual equality** $PC = DC$
- **If** $C_1 \cap \Theta_1(\text{dom}(\Phi_\hat{y})) \neq \emptyset$, **PC has a unique solution in** $L_\eta(\mathbb{P})$
- **If moreover** $C_1 \cap \text{icor}(\Theta_1(\text{dom}(\Phi_\hat{y}))) \neq \emptyset$ **the solution of PC is given by**

$$\hat{Q} = \frac{d\eta^*}{dz}(<\tilde{g}, \theta>)d\mathbb{P}. $$

*where $\tilde{g}$ solves the (extended) dual.*

Here, $\text{icor}(A) = \{a \in A|\forall x \in \text{aff}(A), \exists t > 0 \text{ tq. } a + t(x - a) \in A\}$. 
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Conclusions, open problems
Consider on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t=0}^T, \mathbb{P})\), and for \(t \leq T\), the diffusion
\[
    dS_t = S_t\{b dt + \sigma dW_t\}
\]
\[
    S_0 = 1
\]
(most elementary Black-Scholes).

This model is complete, with risk neutral measure
\[
d\mathbb{P}^* = \exp\left\{-\frac{b}{\sigma} W_T - \frac{b^2}{2\sigma^2} T\right\} d\mathbb{P}.\]
Under \(\mathbb{P}^*\),
\[
    S_T \sim \text{lognormal}\left(-\frac{\sigma^2}{2} T, \sigma^2 T\right).
\]
Example

- We take \( U(x) = 2\sqrt{x}, \ x \in (0, \infty), \) thus \( L_\eta = L^2. \)
- For \( A \geq 0, \) consider the uncertainty set

\[
Q = \{ Q \ll P : \mathbb{E}^Q(S_T) \geq A \}
\]

which is not closed in \( L^0 \) and not bounded in \( L^2, \) but weakly closed in \( L^2. \)

- Constraint qualification condition is met if \( e^{\sigma^2 T} > A > 1 \)
Solution

From the dual representation of the corresponding $\Phi$, it follows:

$$
\inf_{Q \in \mathcal{Q}} \mathbb{E}^P \left[ \frac{dQ}{dP} V \left( y Y_T \frac{dP}{dQ} \right) \right] = \sup_{\mathbb{R}^2} \left[ z_1 + A z_2 - \frac{y}{4} \mathbb{E}^P \left( (z_1 + S_T z_2)^2 1_{z_1 + S_T z_2 > 0} \right) \right]
$$

Right-hand side can be solved, and by means if the duality relation between $u$ and $v$, we get:

$$
u(x) = 2 \sqrt{x \left( 1 + \frac{(A - 1)^2}{e^{\sigma^2 T} - 1} \right)},$$

$$\hat{Q}(d\omega) = \frac{e^{\sigma^2 T} - A + S_T (A - 1)}{e^{\sigma^2 T} - 1} \mathbb{P}(d\omega)$$

and

$$\hat{X}_T := x \frac{\left( e^{\sigma^2 T} - A + S_T (A - 1) \right)^2}{\left( e^{\sigma^2 T} - 1 + (A - 1)^2 \right) \left( e^{\sigma^2 T} - 1 \right)}.$$
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Conclusions, open problems
Functional setting and methodology to solve robust problem in complete market beyond model compactness framework is proposed.

Least favorable measure can be explicitly (or numerically) computed when uncertainty is determined by finitely many moment constraints.
Approach not readily extendable to utility functions not bounded from below or defined in \((-\infty, \infty)\)

Incomplete case: Dual problem can be embedded in a convex modular space with explicit topology (Banach lattice) and dual space. Coercivity holds, but space is not reflexive.
Bibliography I


I. Karatzas, J. Lehoczky, S. Shreve: “Optimal portfolio and consumption decisions for a small investor on a finite horizon” SIAM J. Control Optim. 25 1557-1586


