Selected Problems
Universality Trimester, HIM (Fall, 2013)

Gregory CHERLIN: Problems on the Existence of Countable Universal Objects with Constraints

Problem 1

Let $\mathcal{F}$ be a finite set of forbidden permutation patterns, and let $\neg\mathcal{F}$ denote the corresponding pattern avoidance class of permutations (omitting the forbidden patterns). When is there a universal countable permutation in the class $\neg\mathcal{F}$?—More particularly, is this an algorithmically decidable question?

Explanation. A permutation is a set equipped with two linear orderings. A permutation pattern is an isomorphism type of permutation. An embedding between two permutations is an injection which respects both orderings. There are good tools for analyzing analogous problems for forbidden graphs, but there is as yet no similar theory for permutation classes. Cameron's *generic permutation* [Ca03] is universal for the unrestricted case. His article is also useful for background, particularly for the translation between permutations and structures. If the theory of universal permutations is at all comparable to the graph theoretic case, then the existence of a universal tournament with constraints would be the exception rather than the rule.

The article [AMR05] is based on the fact that the existence of a universal structure for *finite* structures is the same as joint embedding for the class in question (a highly nontrivial property: there are various ways that joint embedding can occur). But requiring a countable universal object for countable structures should isolate some special cases, which while not typical of the general problem, would for example be expected to have a nice structural Ramsey theory.

For a survey of some algorithmic issues in the subject see [Ru05]. Even the problem of deciding whether a class $\neg\mathcal{F}$ has joint embedding is an open problem!

References

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PROBLEM 2.

Prove or disprove: if C is a finite graph consisting of a complete graph K_n with at most one path attached to each vertex, then there is a universal countable C-free graph.

Explanation. Much is known about the existence of universal C-free graphs where C is an arbitrary finite connected graph. When C contains a single nontrivial block the structure is conjectured to be as above. The existence of such a universal C-free graph can be reinterpreted more concretely as a conjecture about the nonexistence of certain complicated finite graphs, and this appears to be a good way to approach the problem. Our references give the following.

[CT07] a proof of the existence of certain universal graphs which shows the relevant technique;

[Ch11b] an analysis of the graph theoretic content of this particular problem.

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DAVID EVANS: EXTREME AMENABILITY AND INVARIANT TYPES FOR ω-CATEGORICAL STRUCTURES.

Suppose M is an ω-categorical structure and $G = \mathrm{Aut}(M)$. We can consider G acting on the Stone space $X = S(M)$ of 1-types over M: the space of ultrafilters over the parameter-definable subsets of M. If $p \in S(M)$ let G_p denote the stabiliser of p in G.

If G is extremely amenable, then it has a fixed point in every closed G-invariant subset of X. In particular, there is a non-principal type $p \in X$ which is invariant under G. Of course, the same applies for the spaces $S_\sigma(M)$ of types over M in all sorts σ of M^{eq}.

Definition. Say that a countable ω-categorical structure M has property (SA) if $G = \mathrm{Aut}(M)$ has a fixed point in every closed G-invariant subset of $S_\sigma(M)$, for all sorts σ of M^{eq}.

The question arises as to whether every ω-categorical structure N has an ω-categorical expansion M which is extremely amenable: cf. Problem 28 of [1]; the question is also explored in [2]. Here we ask a weaker question:

Question 1. *Does every ω-categorical structure N has an ω-categorical expansion M which satisfies (SA)?*

Of course, a negative answer to this gives an ω-categorical structure N with no ω-categorical EA expansion.

A more concrete approximation to this:

Question 2. *Suppose N is a countable ω-categorical structure and $G =$* $\mathrm{Aut}(N)$. *Is there always a non-isolated type $p \in S(N)$ such that G_p is oligomorphic on N?*

Note that if this has a negative answer, then so does Question 1.

This could also be asked for other classes: for example, homogeneous for a finite relational language.

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[1] M. Bodirsky, M. Pinsker and T. Tsankov, Decidability of Definability. arXiv:1012.2381v4
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YONATAN GUTMAN

WHAT IS THE UNIVERSAL MINIMAL SPACE OF Homeo(Q)?

I would like to repeat an open problem of Vladimir Pestov (Problem 6.4.13 in [P06]; the problem is attributed to Uspenskij, see Question 23 on p. 444 of [Ell07]):

Problem. *What is the universal minimal space of* Homeo(Q)*, the group of homeomorphisms of the Hilbert Cube* $Q = [0, 1]^{\mathbb{N}}$*, equipped with the compact open topology?*

The problem can be traced to the last section of [P98], where it was asked if the natural action of Homeo(Q) on Q is the universal minimal space. In [Usp00] Uspenskij showed this not to be the case by showing that for every topological group G, the action of G on the universal minimal space U_G is not 3-transitive, i.e., there exist triples (a_1, a_2, a_3) and (b_1, b_2, b_3) of distinct points of U_G such that no $g \in G$ satisfies $g(a_i) = b_i$ for $i = 1, 2, 3$.

In [G08], I showed that the action of Homeo(Q) on Q is not 1-transitive by constructing a new minimal action, the *space of connected maximal chains* (see details in the article). It is possible this space is the universal minimal space but this remains open. I think this problem is interesting as it has been open for a while and it seems the Fraïssé limits techniques do not have a bearing on it. Maybe a new approach is waiting to be discovered!

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A.S. KECHRIS, M. SOKIĆ AND S. TODORCEVIC

A HOMOGENEOUS VERSION OF THE DUAL RAMSEY THEOREM

Recall the **Dual Ramsey Theorem** of Graham-Rothschild which asserts the following: Given $k < \ell$ and N there is $m > \ell$ so that for any coloring with N colors of the set of equivalence relations on $\{1, \ldots, m\}$ with k many classes, there is an equivalence relation F with ℓ many classes, such that any coarser than F equivalence relation with k many classes has the same color.

We propose here the following homogeneous version of this theorem. Call an equivalence relation **homogeneous** if all its classes have the same cardinality. Let also $k \ll \ell$ mean that $k < \ell$ and k divides ℓ.

Problem (Homogeneous Dual Ramsey). *Is it true that given N and $k \ll \ell$, there is $m \gg \ell$ such that for any coloring with N colors of the homogeneous equivalence relations on $\{1, \ldots, m\}$ with k many classes, there is a homogeneous equivalence relation F with ℓ many classes, so that all homogeneous equivalence relations coarser than F with k many classes have the same color?*

This problem was raised in the paper [KST] and it is motivated from its applications to the study of the dynamics of the group of automorphisms of a standard measure space, which are developed in this paper.

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Julien MELLERAY: Some problems related to the themes of the HIM trimester "Homogeneity and Universality"

1. The Holmes space

During C. Ward Henson's talks, the Urysohn and Gurarij spaces were prominently featured, and analogies between the two were noted. The Urysohn space is related to another interesting Banach space, via the following theorem due to M.R. Holmes: pick a point $0 \in \mathbb{U}$; then, whenever $\mathbb{U}$ is embedded in a Banach space X in such a way that 0 is mapped to the origin of X, the norm of any linear combination of elements of $\mathbb{U}$ is uniquely determined, via the following formula:

$$\left\| \sum_{i=1}^{n} a_i u_i \right\| = \sup \left\{ \sum a_i f(u_i) \colon f \text{ is } 1-\text{Lipschitz and } f(0)=0 \right\}$$

So, whenever the Urysohn space embeds in a Banach space X, its closed linear span is always isometric to the same separable Banach space, which seems to be now called the *Holmes space*. One might suspect that the Holmes space is isometric to the Gurarij space, but such is not the case; its properties are very poorly understood.

In the spirit of C. Ward Henson's talks, and of I. Ben Yaacov's talk during the workshop on homogeneous structures, the following questions about the Holmes space feel natural (of course these questions are related to each other):

(1) Is the theory of the Holmes space $\aleph_0$-categorical?
(2) Can one describe the distance between types for this theory/ understand the action of the (linear) isometry group of the Holmes space on the Holmes space?
(3) Is the linear isometry group of the Holmes space Roelcke-precompact?

I suspect that the answer to the first and third question on the list above is negative; answering the second problem seems to be a natural approach. Since, in a sense, the Holmes space is a canonical object, it would be interesting to know whether it is possible to describe it using only first-order logic (presumably not).

2. The group of isometries of the Gurarij space

As mentioned above, the Gurarij space appears to be an analogue, in the setting of separable Banach spaces, of the Urysohn space; it is a universal separable Banach space, its group of linear isometries (denoted by G below) is a universal Polish group, and the Gurarij space is approximately ultrahomogeneous. Note that no universal separable Banach space can be exactly ultrahomogeneous, as its unit sphere must contain points at which the norm is differentiable and points at which the norm is not differentiable, and two such points cannot be mapped one onto the other by a linear isometry. This lack of exact homogeneity seems to be an obstacle preventing us from applying our current techniques, so that many properties known to hold for other (apparently?) comparable Polish groups are open for G. Let me just mention three of those:

(1) Does G have the automatic continuity property?
(2) Is it true that conjugacy classes in G are meager?
(3) Is G an extremely amenable Polish group?

At the moment, the answer to all three questions above seems likely to be positive. An interesting related problem is the following: is it true that the set of elements of finite order is dense in G? Or, at least, that the set of elements generating a relatively compact group is dense in G? This is really a problem about finite-dimensional normed vector spaces, but apparently this type of question has not been studied a lot. Strengthenings of this property (if true) would imply a positive answer to the second and third questions above, but even this basic question seems difficult to answer.

Since I mentioned that the lack of exact ultrahomogeneity appeared to be an obstacle, another problem presents itself: is it possible to build an exactly ultrahomogeneous Polish metric structure $\mathcal{M}$ such that G is isomorphic (as a topological group) to the automorphism group of $\mathcal{M}$? More generally, can one characterize Polish groups which can be realized as the automorphism group of an exactly ultrahomogeneous Polish metric structure? Is this true of all Polish groups? The isometry group of the Gurarij space seems like a good test case for this question.

André NIES: Questions for HIM workshop Universality and Homogeneity

Polish metric spaces and the classical Scott analysis. A metric space (M, d) can be turned into a structure in the language with binary relations S_q for $q \in \mathbb{Q}^+$, where $S_q(a, b)$ holds if $d(a, b) < q$.

The *Scott rank* $\mathrm{sr}(M)$ of a structure M is defined as the smallest α such that $\equiv_\alpha$ implies $\equiv_{\alpha+1}$ for all tuples of that structure. We remark that always $\mathrm{sr}(M) < |M|^+$.

Friedman, Körwien and Nies (2012) showed that for each $\alpha < \omega_1$, there is an countable Polish ultrametric space M such that $\mathrm{sr}(M) = \alpha \times \omega$.

Question 1.
(*a*) *Does every Polish metric space have countable Scott rank?*
(*b*) *Can it in fact be described within the class of Polish metric spaces by an* $L_{\omega_1,\omega}$ *sentence?*

Reference: Nies' HIM talk on October 9, available at
`http://dl.dropbox.com/u/370127/talks/2013/Nies_HIM_PolishSpaces.pdf`

Borel structures. Given a Borel signature, a Borel structure is one such that the atomic diagram is Borel. It is injective if equality of the structure is represented by true equality on the underlying Polish space.

Examples:

1. Vector space over $\mathbb{R}$ of countable (or perfect) dimension. These have injective presentations.

2. The Boolean algebra $\mathcal{B} = \mathcal{P}(\omega)/\mathrm{fin}$ of subsets of ω modulo finite sets.

Khoussainov, Hjorth, Montalban, and Nies (2008) showed that the Borel structure consisting of the disjoint sum of the Boolean algebras $P(\omega)$, $\mathcal{B}$, and the canonical projection operation has no injective presentation.

Question 2. *Does the Boolean algebra* $\mathcal{B}$ *have an injective Borel presentation? Is this at least consistent with ZFC?*

Discussions with Stevo Todorcevic hinted at a negative answer (in ZFC) to the first question.

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SIMON THOMAS

REPRESENTATION UNIVERSAL GROUPS

Let G be a countable group. Then, by Thoma [1], G has an infinite dimensional irreducible unitary representation if and only if G is not abelian-by-finite. In this case, G is said to be a *non-type I* group. For each countable non-type I group G, let $\mathrm{Irr}_\infty(G)$ be the standard Borel space of infinite dimensional irreducible unitary representations of G and let $\approx_G$ be the corresponding unitary equivalence relation on $\mathrm{Irr}_\infty(G)$.

Definition. The countable non-type I group G is *representation universal* if whenever H is a countable non-type I group, then $\approx_H$ is Borel reducible to $\approx_G$.

For example, it is easily seen that the free group $\mathbb{F}_\infty$ on countably many generators is representation universal.

Conjecture 1. *There exists a countable non-type I group G which is not representation universal.*

By Thomas [2], if G, H are countable non-type I groups and G is amenable, then $\approx_G$ is Borel reducible to $\approx_H$. Thus Conjecture 1 is equivalent to the following:

Conjecture 2. *If G is a countable amenable non-type I group, then G is not representation universal.*

The most basic questions concerning representation universal groups remain open, including the following:

Conjecture 3. *If the countable group G contains a free nonabelian subgroup, then G is representation universal.*

Question. *Does every representation universal group contain a free nonabelian subgroup?*

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Lionel Nguyen VAN THÉ:

Existence of Ramsey precompact expansions

For a Fraïssé structure **F** with language L, a *precompact expansion* of **F** is an expansion $\mathbf{F}^*$ in a language $L^* \supset L$ in the usual sense of first order structures, with the additional requirements that $L^* \setminus L$ is relational, and every element of the age of **F** admits finitely many expansions in the age of **F**.

Question 1. *Let* ***F*** *be an ω-categorical Fraïssé structure. Then it admit a Fraïssé precompact expansion* $\boldsymbol{F}^*$ *whose age satisfies the Ramsey property.*

Explanation: This question appears naturally when trying to understand which classes of finite structures are "close" to being Ramsey. If "close" is understood as "up to precompact expansion", many Fraïssé classes turn out to have that property. For example, this includes numerous classes of metric spaces, all Fraïssé classes of graphs, as well as all classes of directed graphs. How general a phenomenon is that? Note that a positive answer to Question 1 would take the complete opposite direction of the intuition developed after the knowledge accumulated in the seventies and the eighties, which suggests that the Ramsey property is very restrictive, and as such remains exceptional. Another context where Question 1 appears is topological dynamics. This connection builds on the Kechris-Pestov-Todorcevic correspondence developed in [3], where structural Ramsey theory is connected to universal minimal flows of topological groups. A positive answer to Question 1 is equivalent to the fact that every oligomorphic closed subgroup of S_∞ has a metrizable universal minimal flow with a generic orbit (the proof will appear in a forthcoming paper in collaboration with Julien Melleray and Todor Tsankov). Therefore, because oligomorphic subgroups of S_∞ are cases of Roelcke precompact groups, Question 1 is a particular instance of:

Question 2. *Let G be a Roelcke precompact Polish group. Is its universal minimal flow metrizable with a generic orbit?*

Note also the following variant of Question 1, which is of interest because of its appearance in the recent works of Bodirsky-Pinsker [1] in relationship with Thomas' conjecture, and Bodirsky-Pinsker-Tsankov [2] on certain decidability problems.

Question 3. *Same as Question 1, where* ***F*** *and* $\boldsymbol{F}^*$ *are required to have finite relational language.*

To all of those questions, I conjecture a positive answer.

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A. M. VERSHIK: Continuous Universal Structures

1. Urysohn's universal metric space U as a group

After a result of P. Cameron & A. Vershik [CV06] which showed that Urysohn space could be equipped with a structure of an abelian monothetic group, many new problems appeared.

Problem 1.1. *That structure is not unique, and there is a continuum of such structures which are not mutually isometric. But are they isomorphic as abstract groups?*

Problem 1.2. *A. Shkarin, P. Niemic, and M. Doucha* [Sh99, Nie09, Do14a] *introduced another group structure on **U** making it universal in the class of all abelian groups with invariant separable metric, which differs from the Cameron-Vershik (CV) structures in 1.1. Nevertheless, what kind of relations exist between them? For example, because of universality CV-structures must be isometrically embedded in that universal structure.*

Problem 1.3. *What kind of* non-abelian *universal structures with invariant metric can be defined on **U**? What is the connection with Hall's universal locally finite group?*

2. Continuous models and measures

After the paper by A.Petrov & A.Vershik [PV10] about existence of many measures on the space of all universal K_n-free graphs which are invariant with respect to the group of all permutations of the vertices, Ackerman, Freer, and Patel [AFP12] used the same idea (of *continuous* universal graphs) to prove that the same is true for general R. Fraïsse-systems with one additional condition: the strong amalgamation property. These results and the recent study by L. Lovasz & B. Szegedy [LS06] of continuous graphs gives rise to the general problem: how to define Borel or Measurable continuous objects and in particular universal continuous objects?

Problem 2.1. *For definiteness let us consider the category (Fraisse system) of infinite graphs (all or triangle free etc.), and suppose that there exists a universal object in the usual sense. Define a continuous analog of that category in which the object is a symmetric* 0–1*-valued function of two variables with appropriate properties on the standard Borel space. By definition the universal continuous object is a function which satisfied to the same conditions as the countable object.*

What kind of uniqueness we can assume for universal continuous object? It is clear that random countable sub-objects of different universal continuous objects are isomorphic. How to distinguish universal continuous structures?

Problem 2.2. *Vershik's version of Aldous' theorem means that the random universal graph is a complete metric invariant of the function of several arguments, which is a measure-theoretical (continuous) universal graph. How to characterize and classify Borel universal graphs? We can put the same question for hypergraphs and for any Fraisse system.*

Problem 2.3. *In the paper by Petrov & Vershik an invariant measure was constructed on the set of all triangle free graphs. It is well-known that the uniform measures on the set of all triangle-free graphs with* n *vertices is weakly convergent when* n *tends to infinity to the universal bipartite graph. How may we approximate invariant measures on the set of all universal triangle free graphs with the measures on finite triangle-free graphs?*

As Professor Gregory Cherlin has informed me, there are well developed theories of continuous analogs of Fraïssé theory developed by Ben Yaacov, Kubis, and Schoretsanitis [BY13, Ku13, Sch07, see also [KS13] and [MT11]]. It is important to make a link between those results and the more specific results of [PV10]: for example, in what sense is our construction [PV10] (or [AFP12]) of the universal Borel continuous graph (or 3-free universal continuous graph, etc) unique?

References

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